UM

WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

1. The concept of cardinal number. Cantor's continuum problem is simply the question: How many points are there on a straight line in Euclidean space? In other terms, the question is: How many different sets of integers do there exist?

This question, of course, could arise only after the concept of "number" had been extended to infinite sets; hence it might be doubted if this extension can be effected in a uniquely determined manner and if, therefore, the statement of the problem in the simple terms used above is justified. Closer examination, however, shows that Cantor's definition of infinite numbers really has this character of uniqueness, and that in a very striking manner. For whatever "number" as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (*e.g.*, their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence for two sets A and B it is possible (at least theoretically) to change the properties and relations of each element of A into those of the corresponding element of B, whereby A is transformed into a set completely indistinguishable from B, hence of the same cardinal number. For example, assuming a square and a line segment both completely filled with mass points (so that at each point of them exactly one mass point is situated), it follows owing to the demonstrable fact that there exists a one-to-one correspondence between the points of a square and of a line segment, and, therefore, also between the corresponding mass points, that the mass points of the square can be so rearranged as exactly to fill out the line segment, and vice versa. Such considerations, it is true, apply directly only to physical objects, but a definition of the concept of "number" which would depend on the kind of objects that are numbered could hardly be considered as satisfactory.

So there is hardly any choice left but to accept Cantor's definition of equality between numbers, which can easily be extended to a definition of "greater" and "less" for infinite numbers by stipulating that the cardinal number M of a set A is to be called less than the cardinal number N of a set B if M is different from N but equal to the cardinal number of some subset of B. On the basis of these definitions it becomes possible to prove that there exist infinitely many different infinite cardinal numbers or "powers," and that, in particular, the number of subsets of a set is always greater than the number of its elements; furthermore it becomes possible to extend (again without any arbitrariness) the arithmetical operations to infinite numbers (including sums and products with any infinite

number of terms or factors) and to prove practically all ordinary rules of computation.

But, even after that, the problem to determine the cardinal number of an individual set, such as the linear continuum, would not be well defined if there did not exist some "natural" representation of the infinite cardinal numbers, comparable to the decimal or some other systematic denotation of the integers. This systematic representation, however, does exist owing to the theorem that for each cardinal number and each set of cardinal numbers[1] there exists exactly one cardinal number immediately succeeding in magnitude and that the cardinal number of every set occurs in the series thus obtained.[2] This theorem makes it possible to denote the cardinal number immediately succeeding the set of finite numbers by $\aleph_0$ (which is the power of the "denumerably infinite" sets), the next one by $\aleph_1$, *etc.*; the one immediately succeeding all $\aleph_i$ where i is an integer, by $\aleph_\omega$, the next one by $\aleph_{\omega+1}$, *etc.*, and the theory of ordinal numbers furnishes the means to extend this series farther and farther.

2. The continuum problem, the continuum hypothesis and the partila results concerning its truth obtained so far. So the analysis of the phrase "how many" leads unambiguously to quite a definite meaning for the question stated in the second line of this paper, namely, to find out which one of the $\aleph$'s is the number of points on a straight line or (which is the same) on any other continuum in Euclidean space. Cantor, after having proved that this number is certainly greater than $\aleph_0$, conjectured that it is $\aleph_1$, or (which is an equivalent proposition) that every infinite subset of the continuum has either the power of the set of integers or of the whole continuum. This is Cantor's continuum hypothesis.

But, although Cantor's set theory has now had a development of more than sixty years and the problem is evidently of great importance for it, nothing has been proved so far relative to the question what the power of the continuum is or whether its subsets satisfy the condition just stated, except (1) that the power of the continuum is not a cardinal number of a certain very special kind, namely, not a limit of denumerably many smaller cardinal numbers,[3] and (2) that the proposition just mentioned about the subsets of the continuum is

---

[1] As to the question why there does not exist a set of all cardinal numbers, see footnote 12.

[2] In order to prove this theorem the axiom of choice (see: A. Fraenkel, Einleitung in die Mengenlehre, 3rd ed. Berlin, 1928, p. 288 ff.) is necessary, but it may be said that this axiom is, in the present state of knowledge, exactly as well founded as the system of the other axioms. It has been proved consistent, provided the other axioms are so. (See my paper quoted in footnote 13.) It is exactly as evident as the other axioms for sets in the sense of arbitrary multitudes and, as for sets in the sense of extensions of definable properties, it also is demonstrable for those concepts of definability for which, in the present state of knowledge, it is possible to prove the other axioms, namely, those explained in footnotes 17 and 21.

[3] See F. Hausdorff, Mengenlehre, 1st ed. (1914), p. 68. The discoverer of this theorem. J. König, asserted more than he had actually proved (see Math. Ann. 60 (1904), p. 177).

true for a certain infinitesimal fraction of these subsets, the analytical[4] sets.[5] Not even an upper bound, however high, can be assigned for the power of the continuum. Nor is there any more known about the quality than about the quantity of the cardinal number of the continuum. It is undecided whether this number is regular or singular, accessible or inaccessible, and (except for König's negative result) what its character of confinality[4] is. The only thing one knows, in addition to the results just mentioned, is a great number of consequences of, and some propositions equivalent to, Cantor's conjecture.[6]

This pronounced failure becomes still more striking if the problem is considered in its connection with general questions of cardinal arithmetic. It is easily proved that the power of the continuum is equal to $2^{\aleph_0}$. So the continuum problem turns out to be a question from the "multiplication table" of cardinal numbers, namely, the problem to evaluate a certain infinite product (in fact the simplest non-trivial one that can be formed). There is, however, not one infinite product (of factors >1) for which only as much as an upper bound for its value can be assigned. All one knows about the evaluation of infinite products are two lower bounds due to Cantor and König (the latter of which implies a generalization of the aforementioned negative theorem on the power of the continuum), and some theorems concerning the reduction of products with different factors to exponentiations and of exponentiations to exponentiations with smaller bases or exponents. These theorems reduce[7] the whole problem of computing infinite products to the evaluation of $\aleph_\alpha^{cf(\aleph_\alpha)}$ and the performance of certain fundamental operations on ordinal numbers, such as determining the limit of a series of them. $\aleph_\alpha^{cf(\aleph_\alpha)}$, and therewith all products and powers, can easily be computed[8] if the "generalized continuum hypothesis" is assumed, *i.e.*, if it is assumed that $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for every α, or, in other terms, that the number of subsets of a set of power $\aleph_\alpha$ is $\aleph_{\alpha+1}$. But, without making any undemonstrated assumption, it is not even known whether or not $m<n$ implies $2^m<2^n$ (although it is trivial that it implies $2^m\leqq 2^n$), nor even whether $2^{\aleph_0}<2^{\aleph_1}$.

3. Restatement of the problem on the basis of an analysis of the foundations of set theory and results obtained along these lines. This scarcity of results, even as to the most fundamental questions in this field, may be due to some extent to purely mathematical difficulties; it seems, however (see Section 4 below), that there are also deeper reasons behind it and that a complete solution of

[4] See the list of definitions at the end of this paper.

[5] See F. Hausdorff, Mengenlehre, 3rd ed. (1935), p. 32. Even for complements of analytical sets the question is undecided at present, and it can be proved only that they have (if they are infinite) either the power $\aleph_0$ or $\aleph_1$ or continuum (see; C. Kuratowski, Topologie I, Warszawa-Lwow, 1933, p. 246.)

[6] See W. Sierpinski, Hypothese du Continu, Warsaw, 1934.

[7] This reduction can be effected owing to the results and methods of a paper by A. Tarski published in Fund. Math. 7 (1925), p. 1.

[8] For regular numbers $\aleph_\alpha$ one obtains immediately:

$$\aleph_\alpha^{cf(\aleph_\alpha)} = \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

these problems can be obtained only by a more profound analysis (than mathematics is accustomed to give) of the meanings of the terms occurring in them (such as "set," "one-to-one correspondence," *etc.*) and of the axioms underlying their use. Several such analyses have been proposed already. Let us see then what they give for our problem.

First of all there is Brouwer's intuitionism, which is utterly destructive in its results. The whole theory of the $\aleph$'s greater than $\aleph_1$ is rejected as meaningless.[9] Cantor's conjecture itself receives several different meanings, all of which, though very interesting in themselves, are quite different from the original problem, and which lead partly to affirmative, partly to negative answers;[10] not everything in this field, however, has been clarified sufficiently. The "half-intuitionistic" standpoint along the lines of H. Poincare and H. Weyl[11] would hardly preserve substantially more of set theory.

This negative attitude towards Cantor's set theory, however, is by no means a necessary outcome of a closer examination of its foundations, but only the result of certain philosophical conceptions of the nature of mathematics, which admit mathematical objects only to the extent in which they are (or are believed to be) interpretable as acts and constructions of our own mind, or at least completely penetrable by our intuition. For someone who does not share these views there exists a satisfactory foundation of Cantor's set theory in its whole original extent, namely, axiomatics of set theory, under which the logical system of Principia Mathematica (in a suitable interpretation) may be subsumed.

It might at first seem that the set theoretical paradoxes would stand in the way of such an undertaking, but closer examination shows that they cause no trouble at all. They are a very serious problem, but not for Cantor's set theory. As far as sets occur and are necessary in mathematics (at least in the mathematics of today, including all of Cantor's set theory), they are sets of integers, or of rational numbers (*i.e.*, of pairs of integers), or of real numbers (*i.e.*, of sets of rational numbers), or of functions of real numbers (*i.e.*, of sets of pairs of real numbers), *etc.*; when theorems about all sets (or the existence of sets) in general are asserted, they can always be interpreted without any difficulty to mean that they hold for sets of integers as well as for sets of real numbers, *etc.* (respectively, that there exist either sets of integers, or sets of real numbers, or . . . *etc.*, which have the asserted property). This concept of set, however, according to which a set is anything obtainable from the integers (or some other well defined

---

[9] See L. E. J. Brouwer, Atti del IV Congresso Internazionale dei Matematici (Roma 1908), p. 569.

[10] See L. E. J. Brouwer, Over de gondslagen der wiskunde (Amsterdam and Leipzig, 1907) I, 9; III, 2.

[11] See H. Weyl, Das Kontinuum, 2nd ed. 1932. If the procedure of construction of sets described there (p. 20) is iterated a sufficiently large (transfinite) number of times, one gets exactly the real numbers of the model for set theory spoken of below in Section 4, in which the continuum-hypothesis is true. But this iteration would hardly be possible within the limits of the half intuitionistic standpoint.

objects) by iterated application[12] of the operation "set of,"[13] and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naïve" and uncritical working with this concept of set has so far proved completely self-consistent.[14]

But, furthermore, the axioms underlying the unrestricted use of this concept of set, or, at least, a portion of them which suffice for all mathematical proofs ever produced up to now, have been so precisely formulated in axiomatic set theory[15] that the question whether some given proposition follows from them can be transformed, by means of logistic symbolism, into a purely combinatorial problem concerning the manipulation of symbols which even the most radical intuitionist must acknowledge as meaningful. So Cantor's continuum problem, no matter what philosophical standpoint one takes, undeniably retains at least this meaning: to ascertain whether an answer, and if so what answer, can be derived from the axioms of set theory as formulated in the systems quoted.

Of course, if it is interpreted in this way, there are (assuming the consistency of the axioms) *a priori* three possibilities for Cantor's conjecture: It may be either demonstrable or disprovable or undecidable.[16] The third alternative (which is only a precise formulation of the conjecture stated above that the difficulties of the problem are perhaps not purely mathematical) is the most likely, and to seek a proof for it is at present one of the most promising ways of attacking the problem. One result along these lines has been obtained already, namely, that Cantor's conjecture is not disprovable from the axioms of set theory, provided that these axioms are consistent (see Section 4).

It is to be noted, however, that even if one should succeed in proving its undemonstrability as well, this would (in contradistinction, for example, to the proof for the transcendency of π) by no means settle the question definitively.

---

[12] This phrase is to be understood so as to include also transfinite iteration, the totality of sets obtained by finite iteration forming again a set and a basis for a further application of the operation "set of."

[13] The operation "set of x's" cannot be defined satisfactorily (at least in the present state of knowledge), but only be paraphrased by other expressions involving again the concept of set, such as: "multitude of x's," "combination of any number of x's," "part of the totality of x's"; but as opposed to the concept of set in general (if considered as primitive) we have a clear notion of this operation.

[14] It follows at once from this explanation of the term "set" that a set of all sets or other sets of a similar extension cannot exist, since every set obtained in this way immediately gives rise to further application of the operation "set of" and, therefore, to the existence of larger sets.

[15] See, *e.g.*, P. Bernays, A system of axiomatic set theory, J. Symb. Log. 2 (1937), p. 65; 6 (1941), p. 1; 7 (1942), p. 65; p. 133; 8 (1943), p. 89. J. von Neumann, Eine Axiomatisierung der Mengenlehre, J. reine u. angew. Math. 154 (1925), p. 219; cf also: *ibid.*, 160 (1929), p. 227; Math. Zs 27 (1928), p. 669. K. Gödel, The Consistency of the Continuum Hypothesis (Ann. Math. Studies No. 3), 1940.

[16] In case of the inconsistency of the axioms the last one of the four *a priori* possible alternatives for Cantor's conjecture would occur, namely, it would then be both demonstrable and disprovable by the axioms of set theory.

Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to point out ways in which a decision of the question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained.

For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set[17] on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of." These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers or (which is the same) of sets having these cardinal numbers. The simplest of these strong "axioms of infinity" assert the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $> \aleph_0$. The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes).[18] Other axioms of infinity have been formulated by P. Mahlo.[19] Very little is known about this section of set theory, but at any rate these axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of the series of those set up so far.

That these axioms have consequences also far outside the domain of very great transfinite numbers, which are their immediate object, can be proved; each of them (as far as they are known) can, under the assumption of consistency, be shown to increase the number of decidable propositions even in the field of Diophantine equations. As for the continuum problem, there is little hope of solving it by means of those axioms of infinity which can be set up on the basis of principles known today (the above-mentioned proof for the undisprovability of the continuum hypothesis, *e.g.*, goes through for all of them without any change). But probably there exist others based on hitherto unknown principles; also there may exist, besides the ordinary axioms, the axioms of infinity and

---

[17] Similarly also the concept "property of set" (the second of the primitive terms of set theory) can constantly be enlarged, and furthermore concepts of "property of property of set" *etc.* be introduced whereby new axioms are obtained, which, however, as to their consequences for propositions referring to limited domains of sets (such as the continuum hypothesis) are contained in the axioms depending on the concept of set.

[18] See E. Zermelo, Fund. Math. 16 (1930), p. 29.

[19] See: Ber. Verh. Sächs. Ges. Wiss. 63 (1911), pp. 190–200; 65 (1913), pp. 269–276. From Mahlo's presentation of the subject, however, it does not appear that the numbers he defines actually exist.

the axioms mentioned in footnote 14, other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts.

Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success," that is, its fruitfulness in consequences and in particular in "verifiable" consequences, *i.e.*, consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent owing to the fact that analytical number theory frequently allows us to prove number theoretical theorems which can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.

4. Some observations about the question: In what sense and in which direction may a solution of the continuum problem be expected? But are such considerations appropriate for the continuum problem? Are there really any strong indications for its unsolubility by the known axioms? I think there are at least two.

The first one is furnished by the fact that there are two quite differently defined classes of objects which both satisfy all axioms of set theory written down so far. One class consists of the sets definable in a certain manner by properties of their elements,[20] the other of the sets in the sense of arbitrary multitudes, irrespective of, if, or how they can be defined. Now, before it is settled what objects are to be numbered, and on the basis of what one-to-one correspondences, one could hardly expect to be able to determine their number (except perhaps in case of some fortunate coincidence). If, however, someone believes that it is meaningless to speak of sets except in the sense of extensions of definable properties, or, at least, that no other sets exist, then, too, he can hardly expect more than a small fraction of the problems of set theory to be solvable without making use of this, in his opinion essential, characteristic of sets, namely, that they are

[20] Namely, definable "in terms of ordinal numbers" (*i.e.*, roughly speaking, under the assumption that for each ordinal number a symbol denoting it is given) by means of transfinite recursions, the primitive terms of logic, and the ϵ-relation, admitting, however, as elements of sets and of ranges of quantifiers only previously defined sets. See my papers quoted in footnotes 13 and 19, where an exactly equivalent, although in its definition slightly different, concept of definability (under the name of "constructibility") is used. The paradox of Richard, of course, does not apply to this kind of definability, since the totality of ordinals is certainly not denumerable.

all derived from (or in a sense even identical with) definable properties. This characteristic of sets, however, is neither formulated explicitly nor contained implicitly in the accepted axioms of set theory. So from either point of view, if in addition one has regard to what was said above in Section 2, it is plausible that the continuum problem will not be solvable by the axioms set up so far, but, on the other hand, may be solvable by means of a new axiom which would state or at least imply something about the definability of sets.[21]

The latter half of this conjecture has already been verified; namely, the concept of definability just mentioned (which is itself definable in terms of the primitive notions of set theory) makes it possible to derive the generalized continuum hypothesis from the axiom that every set is definable in this sense.[22] Since this axiom (let us call it "A") turns out to be demonstrably consistent with the other axioms, under the assumption of the consistency of these axioms, this result (irrespective of any philosophical opinion) shows the consistency of the continuum hypothesis with the axioms of set theory, provided that these axioms themselves are consistent.[23] This proof in its structure is analogous to the consistency-proof for non-Euclidean geometry by means of a model within Euclidean geometry, insofar as it follows from the axioms of set theory that the sets definable in the above sense form a model for set theory in which furthermore the proposition A and, therefore, the generalized continuum hypothesis is true. But the definition of "definability" can also be so formulated that it becomes a definition of a concept of "set" and a relation of "element of" (satisfying the axioms of set theory) in terms of entirely different concepts, namely, the concept of "ordinal numbers," in the sense of elements ordered by some relation of "greater" and "less," this ordering relation itself, and the notion of "recursively defined function of ordinals," which can be taken as primitive and be described axiomatically by way of an extension of Peano's axioms.[24] [Note that this does not apply to my original formulation presented in the papers quoted above, because there the general concept of "set" with its element relation occurs in the definition of "definable set," although the definable sets remain the same if, afterwards, in the definition of "definability" the term "set" is replaced by "definable set."]

[21] D. Hilbert's attempt at a solution of the continuum problem (see Math. Ann. 95 (1926), p. 161), which, however, has never been carried through, also was based on a consideration of all possible definitions of real numbers.

[22] On the other hand, from an axiom in some sense directly opposite to this one the negation of Cantor's conjecture could perhaps be derived.

[23] See my paper quoted in footnote 13 and note Proc. Nat. Ac. Sci. 25 (1939), p. 220. I take this opportunity to correct a mistake in the notation and a misprint which occurred in the latter paper: in the lines 25 to 29 of page 221, 4 to 6 and 10 of page 222, 11 to 19 of page 223, the letter α should be replaced (in all places where it occurs) by μ. Also, in Theorem 6 on page 222 the symbol "$\equiv$" should be inserted between $\phi_\alpha(x)$ and $\phi_{\bar{\alpha}}(x')$. For a carrying through of the proof in all details the paper quoted in footnote 13 is to be consulted.

[24] For such an extension see A. Tarski, Ann. Soc. Pol. Math. 3 (1924), p. 148, where, however, the general concept of "set of ordinal numbers" is used in the axioms; this could be avoided, without any loss in demonstrable theorems, by confining oneself from the beginning to recursively definable sets of ordinals.

A second argument in favor of the unsolubility of the continuum problem on the basis of the ordinary axioms can be based on certain facts (not known or not existing at Cantor's time) which seem to indicate that Cantor's conjecture will turn out to be wrong;[25] for a negative decision the question is (as just explained) demonstrably impossible on the basis of the axioms as known today.

There exists a considerable number of facts of this kind which, of course, at the same time make it likely that not all sets are definable in the above sense.[26] One such fact, for example, is the existence of certain properties of point sets (asserting an extreme rareness of the sets concerned) for which one has succeeded in proving the existence of undenumerable sets having these properties, but no way is apparent by means of which one could expect to prove the existence of examples of the power of the continuum. Properties of this type (of subsets of a straight line) are: (1) being of the first category on every perfect set,[27] (2) being carried into a zero set by every continuous one-to-one mapping of the line on itself.[28] Another property of a similar nature is that of being coverable by infinitely many intervals of any given lengths. But in this latter case one has so far not even succeeded in proving the existence of undenumerable examples. From the continuum hypothesis, however, it follows that there exist in all three cases not only examples of the power of the continuum,[29] but even such as are carried into themselves (up to denumerably many points) by *every* translation of the straight line.[30]

And this is not the only paradoxical consequence of the continuum hypothesis. Others, for example, are that there exist: (1) subsets of a straight line of the power of the continuum which are covered (up to denumerably many points) by *every* dense set of intervals, or (in other terms) which contain no undenumerable subset nowhere dense on the straight line,[31] (2) subsets of a straight line of the power of the continuum which contain no undenumerable zero set,[32] (3) subsets of Hilbert space of the power of the continuum which contain no undenumerable subset of finite dimension,[33] (4) an infinite sequence A^i of decompositions of any set M of the power of the continuum into continuum

[25] Views tending in this direction have been expressed also by N. Lusin in Fund. Math. 25 (1935), p. 129 ff. See also: W. Sierpinski, *ibid.*, p. 132.

[26] That all sets are "definable in terms of ordinals" if *all* procedures of definition, *i.e.* also quantification and the operation $\hat{x}$ with respect to *all* sets, irrespective of whether they have or can be defined, are admitted could be expected with more reason, but still it would not at all be justified to assume this as an axiom. It is worth noting that the proof that the continuum hypothesis holds for the definable sets or follows from the assumption that all sets are definable, does not go through for this kind of definability, although the assumption that these two concepts of definability are equivalent is, of course, demonstrably consistent with the axioms.

[27] See W. Sierpinski, Fund. Math. 22 (1934), p. 270 and C. Kuratowski, Topologie I, p. 269 ff.

[28] See N. Lusin and W. Sierpinski, Bull. Internat. Ac. Sci. Cracovie 1918, p. 35, and W. Sierpinski, Fund Math. 22 (1934), p. 270.

[29] For the 3rd case see: *l.c.* 6, p. 39, Th. 1.

[30] See W. Sierpinski, Ann. Scuol. Norm. Sup. Pisa 4 (1935), p. 43.

[31] See N. Lusin, C. R. Paris 158 (1914), p. 1259.

[32] See W. Sierpinski, Fund. Math. 5 (1924), p. 184.

[33] See W. Hurewicz, Fund. Math. 19 (1932), p. 8.

many mutually exclusive sets A_x^t such that, in whichever way a set $A_{x_i}^t$ is chosen for each i, $\Pi_i(M-A_{x_i}^t)$ is always denumerable.[34] Even if in (1)–(4) "power of the continuum" is replaced by "$\aleph_1$" these propositions are very implausible; the proposition obtained from (3) in this way is even equivalent with (3).

One may say that many of the results of point-set theory obtained without using the continuum hypothesis also are highly unexpected and implausible.[35] But, true as that may be, still the situation is different there, insofar as in those instances (such as, *e.g.*, Peano's curves) the appearance to the contrary can in general be explained by a lack of agreement between our intuitive geometrical concepts and the set-theoretical ones occurring in the theorems. Also, it is very suspicious that, as against the numerous plausible propositions which imply the negation of the continuum hypothesis, not one plausible proposition is known which would imply the continuum hypothesis. Therefore one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

Definitions of some of the technical terms

Definitions 4–12 refer to subsets of a straight line, but can be literally transferred to subsets of Euclidean spaces of any number of dimensions; definitions 13–14 refer to subsets of Euclidean spaces.

1. I call "character of confinality" of a cardinal number m (abbreviated by "$cf(m)$") the smallest number n such that m is the sum of n numbers $<m$.
2. A cardinal number m is regular if $cf(m)=m$, otherwise singular.
3. An infinite cardinal number m is inaccessible if it is regular and has no immediate predecessor (*i.e.*, if, although it is a limit of numbers $<m$, it is not a limit of fewer than m such numbers); it is inaccessible in the stronger sense if each product (and, therefore, also each sum) of fewer than m numbers $<m$ is $<m$. [See: W. Sierpinski and A. Tarski, Fund. Math. 15 (1930), p. 292; A. Tarski, Fund. Math. 30 (1938), p. 68. From the generalized continuum hypothesis follows the equivalence of these two notions. This equivalence, however, is a much weaker and much more plausible proposition. $\aleph_0$ evidently is inaccessible in both senses. As for finite numbers, 0 and 2 and no others are inaccessible in the stronger sense (by the above definition), which suggests that the same will hold also for the correct extension of the concept of inaccessibility to finite numbers.]
4. A set of intervals is dense if every interval has points in common with some interval of the set. [The end-points of an interval are not considered as points of the interval.]
5. A zero-set is a set which can be covered by infinite sets of intervals with arbitrarily small lengths-sum.
6. A neighborhood of a point P is an interval containing P.
7. A subset A of B is dense in B if every neighborhood of any point of B contains points of A.
8. A point P is in the exterior of A if it has a neighborhood containing no point of A.
9. A subset A of B is nowhere dense on B if those points of B which are in the exterior of A are dense in B. [Such sets A are exactly the subsets of the borders of the open sets in B, but the term "border-set" is unfortunately used in a different sense.]
10. A subset A of B is of the first category in B if it is the sum of denumerably many sets nowhere dense in B.

[34] See S. Braun and W. Sierpinski, Fund. Math. 19, (1932), p. 1, proposition (Q). This proposition and the one stated under (3) in the text are equivalent with the continuum hypothesis.

[35] See, *e.g.*, L. Blumenthal, Am. Math. Monthly 47 (1940), p. 346.

11. Set A is of the first category on B if the intersection $A \cdot B$ is of the first category in B.
12. A set is perfect if it is closed and has no isolated point (*i.e.*, no point with a neighborhood containing no other point of the set).
13. Borel-sets are defined as the smallest system of sets satisfying the postulates:
    (1) The closed sets at Borel-sets.
    (2) The complement of a Borel-set is a Borel-set.
    (3) The sum of denumerably many Borel-sets is a Borel-set.
14. A set is analytic if it is the orthogonal projection of some Borel-set of a space of next higher dimension. [Every Borel-set therefore is, of course, analytic.]
15. Quantifiers are the logistic symbols standing for the phrases: "for all objects x" and "there exist objects x." The totality of objects x to which they refer is called their range.
16. The symbol "$\hat{x}$" means "the set of those objects x for which $\cdots$"

PERSONNEL AND TRAINING PROBLEMS IN STATISTICS

S. S. WILKS, Princeton University

1. Introduction. A report* has been prepared by the Committee** on Applied Mathematical Statistics of the National Research Council to: (1) analyze some factors contributing to the recent extraordinary growth of interest in the use of statistical methods, (2) present some information on the current and future needs of statistically trained personnel, (3) examine the impact of these needs on present training facilities, and (4) indicate some steps which might establish a training program adequate to meet these growing needs. This is a summary of the report.

2. Statistical organizations. As a simple indication of growth of interest in statistical methods, the Committee describes the formation and recent growth of statistical organizations. The American Statistical Association, founded more than 100 years ago, had a membership of 1700 in 1935. By the end of 1946 it had nearly 4000 members. The Institute of Mathematical Statistics, formed in

* "Personnel and Training Problems Created by the Recent Growth of Applied Statistics in the United States," a report by the Committee on Applied Mathematical Statistics, National Research Council, Washington, D. C., NRC Reprint and Circular Series No. 128, May, 1947, 17 pp.

** Luther P. Eisenhart, Chairman, Former Chairman Division of Physical Sciences, National Research Council
Samuel S. Wilks, Secretary, Professor of Mathematical Statistics, Princeton University
Chester I. Bliss, Biometrician, Connecticut Agricultural Experiment Station
Edward U. Condon, Director, National Bureau of Standards
Harold O. Gulliksen, Professor of Psychology, Princeton University
Lowell J. Reed, Vice-President of the University, and Professor of Biostatistics, School of Hygiene and Public Health, The Johns Hopkins University
Charles F. Roos, President, The Econometric Institute, Inc.
Walter A. Shewhart, Research Engineer, Bell Telephone Laboratories
Hugh M. Smallwood, Director, Department of Physical Research, General Laboratories, U. S. Rubber Company
Frederick F. Stephan, Professor of Sociology and Statistics, Cornell University

1935 to promote the development of statistical theory, had 900 members by the end of 1946. The Econometric Society with a membership of more than 750 was organized in 1930 to promote the application of mathematics and statistical methods in economics. The Psychometric Society is a similar organization for psychology. It was organized in 1935 and now has more than 200 members. The Biometrics Section of the American Statistical Association was formed in 1938 for sponsoring similar work in the biological sciences. It now has more than 1100 members. The most recent statistical organization is the American Society for Quality Control which is concerned with applications of statistical methods in industry. It was organized early in 1946 and now has approximately 2000 members, mostly engineers of various kinds. There are other organizations with considerable interest in statistical methods such as the American Marketing Association, American Public Health Association, American Sociological Society, and Population Association of America.

3. Demand for statisticians. According to the report, there is a heavy demand for both academic and non-academic statistical personnel. The non-academic fields which account for most of the recent growth of interest in statistical methods are: (1) industrial statistical control (in quality control, research and development), (2) research in the biological sciences, (3) collection and analyses of government statistics, (4) market research and commercial surveys, and (5) psychological testing. The Committee, in its report, discusses each of these fields in some detail. Demands are continuing and increasing for statistical personnel in some of the older fields such as finances and taxes, labor and employment, prices and production. Demands for more statistical training for social scientists are increasing.

The Committee made an inquiry among 30 leading authorities at 27 universities in mathematical and applied statistics as to requests received for statistical personnel for a period of approximately six months after the end of the war. These authorities reported a total of 135 requests for personnel for academic positions in mathematical and applied statistics ranging from instructorships to full professorships. No attempt was made to have each respondent identify each request so as to eliminate duplication. But one person reported that he had requests from 21 college and university mathematics departments for Ph.D.'s in mathematical statistics. Another reported 12 requests for Ph.D.'s in agronomy with minors in statistics. There were reported 90 requests from government agencies and 140 requests from industry. The training requirements for these requests ranged from B.A.'s to Ph.D.'s in mathematical and applied statistics.

At least a rough comparison may be made between demands for personnel in mathematics, physics, and statistics. As of December 31, 1945, the National Roster of Scientific and Specialized Personnel had registrations of 9972, 9608 and 2018 in mathematics, physics, and statistics, respectively. For the period September 1, 1945, to May 31, 1946, the number of requests for personnel in these three fields per 1000 persons registered were 4.4, 23.9, and 30.7, respectively.

4. Training in statistics. The Committee devoted more than a third of its report to problems of education and training in statistics. These problems were discussed at two levels: (1) the undergraduate level and (2) the graduate level. It was stated that substantial progress had been made in the teaching of statistics at the graduate level in a number of universities; but that it was still inadequate to meet the growing demands for statistical personnel. The Committee charged that the teaching of statistics at the undergraduate level is still in a very chaotic condition. Graduate teaching in mathematical statistics is more standardized than that in applied statistics. The Committee listed basic requirements in mathematics for graduate training in mathematical statistics as follows: real and complex variables, linear and quadratic forms, matrix algebra, n-dimensional euclidean geometry, measure and integration theory. The courses are essential for the theory of probability which is the foundation for courses in advanced mathematical statistics covering distribution theory, estimation theory, testing of statistical hypotheses and multivariate statistical theory.

Of the 27 universities included in the Committee's inquiry, only ten claimed a graduate program leading to a Ph.D. degree in mathematical statistics, and fourteen claimed an adequate training program at the advanced level for some field of applied statistics. It was found that only four of the universities have special stipends for graduate work in mathematical statistics. The situation in applied statistics is hardly more adequate.

In its discussion of the teaching of elementary statistics the Committee emphasized the duplication of material in elementary statistics courses as they are now taught in various departments of a given college or university, as well as the heterogeneity of the quality of teaching. The Committee expressed its opinion that the standardization and improvement of the teaching of statistics at the undergraduate level is a basic requirement for the solution to the problem of training statistical personnel. Specifically, it proposes that there should be developed a basic course in statistics at the freshman level for students who will go into the natural and social sciences, standardized to the same extent as introductory courses in mathematics, physics, and chemistry.

According to the Committee, one of the most puzzling problems regarding statistics is how it should be organized within a university. Two plans which are being tried out at certain universities were discussed: (1) the statistical laboratory and research center which would serve as an informal campus statistical center, and (2) the department of statistics. Plan (1) is necessarily rather informal and depends for its success on the voluntary cooperation of personnel from various departments who are interested in research and teaching of statistics. Plan (2) would be more formal and desirable but its success would depend on joint membership of some of its personnel with other departments. This is particularly important for the effective teaching of applied statistics which should be carried out in conjunction with departments interested in applications of statistical methods.

5. Summary. The Committee summarized its conclusions as follows:

1. There should be developed a basic introductory course in statistics at preferably the freshman level for colleges and universities throughout the country.

2. The laboratory work in the average course in statistics is inadequate, particularly at the elementary level; experimental work should replace much of the computation at this level.

3. The minimum requirement in effective organization is a central statistical laboratory with which all of those teaching or doing research in statistics would be associated, even though informally in some cases.

4. More success is to be expected from a department of statistics associated with a statistical laboratory, and having some members in common with other departments.

5. The number of institutions needed for giving first-class training through the graduate level are: (a) 5 to 10 in mathematical statistics, (b) 25 to 30 in varying fields of applied statistics.

6. An institution giving complete training in either mathematical or applied statistics should give some training in the other.

7. Institutional stipends for graduate students specializing in mathematical and applied statistics are inadequate.

8. In strengthening its statistical work at the advanced and research levels any given university should consider which field it can develop most effectively, so as to avoid duplication and inefficiency from a national point of view.

9. The immediate critical shortage of highly qualified teachers can be eased only by suitable training of high-grade personnel now in fields of application, or mathematics.

10. There should be an adequate number of postdoctoral fellowships in statistics.

11. Arrangements should be established whereby postgraduate students, research workers, and teachers on leave would be able to obtain work experience in certain government agencies, industrial laboratories, and business research organizations.

12. To help offset the present critical shortage of qualified personnel in applied statistics, it would be desirable to promote conferences at advanced levels and short courses at the elementary level in various fields.

A SIMPLE ANALYTIC PROOF OF A GENERAL χ^2 THEOREM

C. L. WEAVER, Boston, Mass.

Many of the applications of the χ^2 distribution can be justified as particular cases of a general theorem. It is the purpose of this paper to state and give a simple analytic proof of this theorem. Two of the more basic χ^2 principles are stated as corollaries of the general theorem.

The usual procedure in developing the applications of χ^2 is to treat each application more or less independently. When proofs are included they usually involve geometry in n-space, the moment generating function, the Laplace transform, or the algebra of quadratic forms. The proof given here makes use of none of these methods. It does make use of the Jacobian in the change of variables in a multivariate differential (or multiple integral).

DEFINITION. We shall first define the χ^2 distribution with ν degrees of freedom as that distribution having the probability element

$$K_\nu(\chi^2)^{(\nu/2)-1}e^{-\frac{1}{2}\chi^2}d(\chi^2), \qquad 0 \leqq \chi^2 < \infty.$$

Substituting $\chi^2 = 2w$ and equating the integral of the above differential over the given range of χ^2 to unity, it follows by the definition of the gamma integral that

$$K_\nu = \left\{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right\}^{-1}.$$

Let us consider y_i, $i=1$ to n, to be n independent random variables. Each y_i is normally distributed with mean $a_0+a_1t_{1i}+a_2t_{2i}\cdots+a_lt_{li}$ and variance σ_i^2. The t_{1i} to t_{li} may be functionally related but are assumed to be linearly independent. Of the set of $l+1$ parameters a_j we know the true values of any l_1 (the set a_r) and do not know the values of the other l_2 (the set a_s). It is assumed that $l_2 < n$.

THEOREM. *If the set $\hat{a}_s$ be the maximum likelihood estimates of the set a_s, then*

$$P = \sum_i \frac{1}{\sigma_i^2}\left\{y_i - \sum_r a_r t_{ri} - \sum_s \hat{a}_s t_{si}\right\}^2,$$

is distributed as χ^2 with $n-l_2$ degrees of freedom, and independently of the set $\hat{a}_s$ which are jointly distributed in the l_2-variate normal distribution

$$C_1 e^{-\frac{1}{2}Q}\prod_s d\hat{a}_s$$

with

$$Q = \sum_i \frac{1}{\sigma_i^2}\left\{\sum_s (\hat{a}_s - a_s)t_{si}\right\}^2.$$

This Q is distributed independently of P as χ^2 with l_2 degrees of freedom. If

$$R = \sum_i \frac{1}{\sigma_i^2}\left\{y_i - \sum_j a_j t_{ji}\right\}^2,$$

then $R=P+Q$, and R is distributed as χ^2 with n degrees of freedom.

Proof of the Theorem: To simplify the discussion (and this has already been done in the expressions for P and R in the statement of the theorem) we shall introduce the variable t_{0i} in the expression for the means with the understanding that $t_{0i}=1$.

The joint distribution of the y_i is

$$C_2 e^{-\frac{1}{2}R} dy_1 dy_2 \cdots dy_n$$

and this will be a maximum when R is a minimum. The $\hat{a}_s$ are therefore determined so as to make

$$\sum_i \frac{1}{\sigma^2}\left\{y_i - \sum_r a_r t_{ri} - \sum_s \hat{a}_s t_{si}\right\}^2$$

a minimum.

The l_2 estimates $\hat{a}_s$ are thus determined by the l_2 linear equations

$$\sum_i \frac{t_{si}}{\sigma_i^2}\left\{y_i - \sum_r a_r t_{ri} - \sum_s \hat{a}_s t_{si}\right\} = 0. \tag{1}$$

We next note that

$$y_i - \sum_j a_j t_{ji} = \left\{y_i - \sum_r a_r t_{ri} - \sum_s \hat{a}_s t_{si}\right\} + \left\{\sum_s (\hat{a}_s - a_s) t_{si}\right\},$$

we square both sides, divide by σ_i^2, and sum over i, to find by using (1) that

$$R = P + Q.$$

Now let

$$y_i = \sum_r a_r t_{ri} + \sum_s \hat{a}_s t_{si} + \lambda_i \sqrt{P}, \qquad i = 1 \text{ to } n.$$

By the definition of P and (1) we have the n variables λ_i satisfying the l_2+1 equations

$$\begin{aligned} &\sum_i \frac{\lambda_i^2}{\sigma_i^2} = 1, \\ &\sum_i \frac{\lambda_i t_{si}}{\sigma_i^2} = 0. \end{aligned} \tag{2}$$

We now can write the joint probability element of $y_1, y_2, \cdots, y_n$ as

$$C_2 e^{-\frac{1}{2}P} e^{-\frac{1}{2}Q} dy_1 dy_2 \cdots dy_n. \tag{3}$$

Next we shall change variables in this differential from $y_1, y_2, \cdots, y_n$ to P, the set $\hat{a}_s, \lambda_1, \lambda_2, \cdots, \lambda_{n-l_2-1}$.

The Jacobian of the transformation becomes

$$J_1 = \begin{vmatrix} \frac{\partial y_1}{\partial P} \cdots & \frac{\partial y_1}{\partial \hat{a}_s} \cdots & \frac{\partial y_1}{\partial \lambda_1} \cdots & \frac{\partial y_1}{\partial \lambda_{n-l_2-1}} \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ \frac{\partial y_n}{\partial P} \cdots & \frac{\partial y_n}{\partial \hat{a}_s} \cdots & \frac{\partial y_n}{\partial \lambda_1} \cdots & \frac{\partial y_n}{\partial \lambda_{n-l_2-1}} \end{vmatrix}.$$

We have

$$\frac{\partial y_i}{\partial P} = \frac{\lambda_i}{2\sqrt{P}},$$

$$\frac{\partial y_i}{\partial \hat{a}_s} = t_{si},$$

$$\frac{\partial y_i}{\partial \lambda_k} = 0 \qquad \text{for } k \neq i, \text{ and } = \sqrt{P} \text{ for } k = i.$$

Therefore

$$J_1 = \begin{vmatrix} \frac{\lambda_1}{2\sqrt{P}} \cdots t_{s1} \cdots \sqrt{P} & 0 & \cdots \\ \frac{\lambda_2}{2\sqrt{P}} \cdots t_{s2} \cdots 0 & \sqrt{P} & \cdots \\ \cdots\cdots\cdots\cdots\cdots & \cdots & \cdots \\ \frac{\lambda_n}{2\sqrt{P}} \cdots t_{sn} \cdots 0 & 0 & \cdots \end{vmatrix}$$

$$= P^{(n-l_2)/2-1} g_1(\lambda_1, \lambda_2, \cdots, \lambda_n) = P^{(n-l_2)/2-1} g_2(\lambda_1, \lambda_2, \cdots, \lambda_{n-l_2-1}).$$

To obtain g_2 we use (2) to eliminate λ_{n-l_2} to λ_n from g_1. We now can write (3) as

$$C_2 P^{(n-l_2)/2-1} e^{-\frac{1}{2}P} dP e^{-\frac{1}{2}Q} \prod_s d\hat{a}_s g_2(\lambda_1, \lambda_2, \cdots, \lambda_{n-l_2-1}) d\lambda_1 \cdots d\lambda_{n-l_2-1}.$$

This shows that P, the set $\hat{a}_s$,and the σ variables are three independent sets of variables. Therefore P is distributed as

$$C_3 P^{(n-l_2)/2-1} e^{-\frac{1}{2}P} dP, \tag{4}$$

and the set $\hat{a}_s$ is distributed independently as

$$C_1 e^{-\frac{1}{2}Q} \prod_s d\hat{a}_s. \tag{5}$$

These are the required χ^2 and normal l_2-variate distributions for P and the set $\hat{a}_s$.

To prove that R is distributed as χ^2 with n degrees of freedom we need only note that R is independent of l_2 and the $\hat{a}_s$., and that P becomes R when $l_2=0$.

The definition of Q in the theorem shows that Q is a function of the set $\hat{a}_s$. Since the latter is distributed independently of P, so is Q.

To find the distribution of Q we start with (5), letting

$$\sum_s (\hat{a}_s - a_s)t_{si} = \delta_i\sqrt{Q}, \qquad i = 1 \text{ to } n \tag{6}$$

and change variables from the l_2 variables $\hat{a}_s$ to $\delta_1, \delta_2, \cdots \delta_{l_2-1}, Q$. If J_2 is the Jacobian of this transformation, then $1/J_2$ is the Jacobian of the inverse transformation.

$$\frac{1}{J_2} = \begin{vmatrix} \frac{\partial\delta_1}{\partial\hat{a}_1} & \cdots & \frac{\partial\delta_{l_2-1}}{\partial\hat{a}_1} & \frac{\partial Q}{\partial\hat{a}_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\delta_1}{\partial\hat{a}_{l_2}} & \cdots & \frac{\partial\delta_{l_2-1}}{\partial\hat{a}_{l_2}} & \frac{\partial Q}{\partial\hat{a}_{l_2}} \end{vmatrix}.$$

We have

$$\frac{\partial\delta_i}{\partial\hat{a}_s} = \frac{t_{si}}{\sqrt{Q}},$$

$$\frac{\partial Q}{\partial\hat{a}_s} = -\sum_i \frac{2}{\sigma_i^2}\left\{\sum_s (\hat{a}_s - a_s)t_{si}\right\} t_{si}$$

$$= -\sqrt{Q}\sum_i \frac{2\delta_i t_{si}}{\sigma_i^2}.$$

Hence

$$\frac{1}{J_2} = \left(\frac{1}{\sqrt{Q}}\right)^{l_2-1}\sqrt{Q}\, g_3(\delta_1, \delta_2, \cdots, \delta_n),$$

and

$$J_2 = Q^{(l_2/2)-1}g_4(\delta_1, \delta_2, \cdots, \delta_{l_2-1}).$$

In going from $1/g_3$ to g_4 we have eliminated δ_{l_2} to δ_n by use of (6) and

$$\sum_i \frac{\delta_i^2}{\sigma_i^2} = 1. \tag{7}$$

The latter follows from using (6) in the expression for Q in the statement of the theorem. It should be noted that (6) consists of n equations in the l_2 variables

$\delta_s - a_s$. It is possible to solve (6) for δ_{l_2+1} to δ_n in terms of δ_1 to δ_{l_2} with the factor $\sqrt{Q}$ cancelling out in the process. Equation (7) then gives the additional relationship to eliminate δ_{l_2}.

Expression (5) may now be written as

$$C_1 Q^{(l_2/2)-1} e^{-\frac{1}{2}Q} dQ g_4(\delta_1, \delta_2, \cdots, \delta_{l_2-1}) d\delta_1 \cdots d\delta_{l_2-1}.$$

This shows that Q is distributed independently of the δ variables as

$$C_4 Q^{(l_2/2)-1} e^{-\frac{1}{2}Q} dQ.$$

The latter is the χ^2 distribution with l_2 degrees of freedom.

It may be noted that the part of the analysis that leads to $R = P + Q$ includes an indirect proof of the theorem that if χ_1^2 and χ_2^2 are distributed independently as χ^2 with ν_1 and ν_2 degrees of freedom, respectively, then $\chi^2 = \chi_1^2 + \chi_2^2$ is distributed as χ^2 with $\nu_1 + \nu_2$ degrees of freedom. Also the part of the theorem concerning the distribution of Q is a direct proof of the theorem concerning the χ^2 distribution with k degrees of freedom of minus two times the exponent in a normal k-variate distribution. The latter would be the special case $n = l_2 = k$.

COROLLARY 1. If u_i, $i = 1$ to n, are n independent normal variables with common mean a and common variance σ^2, then

$$\chi^2 = \frac{1}{\sigma^2} \sum_i (u_i - a)^2$$

is distributed as χ^2 with n degrees of freedom.

COROLLARY 2. If u_i are defined as in Corollary 1 and $n\bar{u} = \sum_i u_i$, then

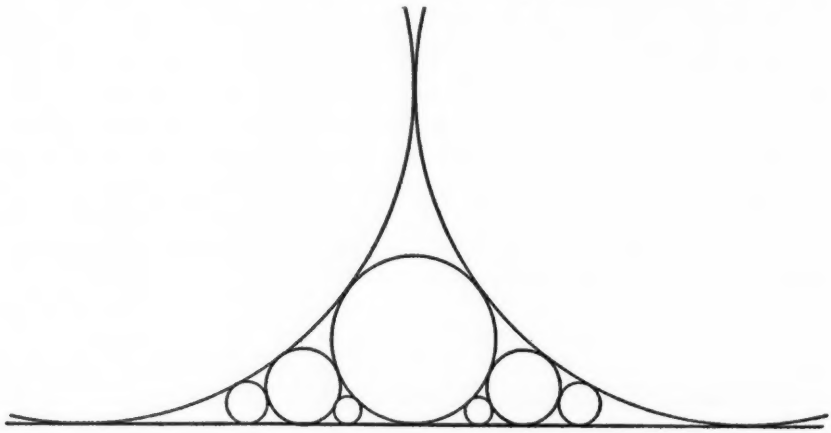
$$\chi^2 = \frac{1}{\sigma^2} \sum_i (u_i - \bar{u})^2$$

is distributed as χ^2 with $n - 1$ degrees of freedom.

A FAMILY OF INTEGERS AND A THEOREM ON CIRCLES

G. T. WILLIAMS, Brookhaven National Laboratory, N. Y.
D. H. BROWNE, Buffalo, N. Y.

1. Introduction. Between two tangent unit circles lying on a line L, a third circle is inscribed tangent to both circles and to L (see figure). Then two more circles are inscribed in the newly-created spaces, each tangent to two circles and to L. Now four circles are inscribed similarly along L, and so on *ad inf.*



We propose to determine the area-sum of circles in this configuration.*

2. The theorems.

THEOREM 1. *If two circles of radii $1/r^2$ and $1/s^2$ are tangent to each other and to L, the radius of the smaller circle tangent to both circles and to L is $1/(r+s)^2$, that of the larger circle being $1/(r-s)^2$.*

This is readily established by comparing the projections of the three lines of centers on L. Thus, in the figure, the first inscribed circle has a radius of $\frac{1}{4}$, the next two have radii of $1/9$, and so on.

DEFINITION. Calling the primary position (the two unit circles) the 0th stage, at the nth stage we have 2^n+1 circles whose radii, reading across, we designate as

$$\{1/A_\nu^n\}^2 \qquad (\nu = 0, 1, \cdots, 2^n),$$

* There is a treatment at some length of these circles in a paper on Fractions by L. R. Ford in this MONTHLY, vol. 45, pp. 586–601 (November, 1938). Some of the results of the present paper will be found there. The problem of the sum of the areas, however, was not considered.—EDITOR.

where the A are successively the positive integers

$$1 \quad 1$$
$$1\ 2\ 1$$
$$1\ 3\ 2\ 3\ 1$$
$$1\ 4\ 3\ 5\ 2\ 5\ 3\ 4\ 1$$
$$1\ 5\ 4\ 7\ 3\ 8\ 5\ 7\ 2\ 7\ 5\ 8\ 3\ 7\ 4\ 5\ 1$$
$$\cdots\cdots\cdots\cdots\cdots$$

These obviously satisfy the relations,

$$A_\nu^n = A_{2^n-\nu}^n,$$
$$A_{2\nu}^{n+1} = A_\nu^n,$$
$$A_{2\nu+1}^{n+1} = A_\nu^n + A_{\nu+1}^n.$$

These numbers have a host of properties, some, involving A_ν^n/A_ν^{n+1}, being akin to properties involved in the theory of Farey Series (*cf.* Hardy & Wright, *Introduction to Theory of Numbers*, p. 23), but we mention only a few which do not bear directly on the principal theorem.

THEOREM 2. *If S_n denotes $\sum_{\nu=0}^{2^n} A_\nu^n$,* then $S_n = 3^n + 1$.

For it is evident that $S_n = S_{n-1} + 2(S_{n-1} - 1) = 3S_{n-1} - 2$.

Since $A_0^n = 1$ for all n, we have immediately, $A_1^n = n+1$. Now, since the remaining A are linear combinations of previous ones, we have an inductive proof that

THEOREM 3. *A_ν^n is a linear function of n for constant ν.*

Indeed, from the recurrence, we find $A_2^n = n$, $A_3^n = 2n-1$, $A_4^n = n-1$, $A_5^n = 3n-4$, and so on. These linear coefficients also behave interestingly, but a discussion of them would carry us too far afield.

THEOREM 4. *For fixed n, the maximum A_ν^n is f_n, the nth Fibonacci number ($\{f_n\}_0^\infty = 1, 2, 3, 5, \cdots$), and is given by $\nu = (2^n - (-1)^n)/3$, and symmetrically.*

THEOREM 5.

$$A_\nu^n \mid A_{\nu-1}^n + A_{\nu+1}^n \qquad \text{(all } n, \nu\text{)}.$$

The proof is inductive. By assuming the proposition for $n-1$ and all ν, we deduce its validity for n and all ν. When the subscript is odd, the theorem holds trivially. When it is even, we observe that

$$A_{2\nu-1}^n + A_{2\nu+1}^n = A_{\nu-1}^{n-1} + 2A_\nu^{n-1} + A_{\nu+1}^{n-1}$$

and by the inductive hypothesis,

$$A_{r}^{n-1} \mid A_{r-1}^{n-1} + A_{r+1}^{n-1}.$$

But

$$A_{r}^{n-1} = A_{2r}^{n}.$$

Hence it is established generally.

We now prove what is the fundamental property of this table of numbers.

THEOREM 6. *Every coupled pair of relatively prime integers occurs in the table of A's.*

Again the proof is inductive and depends on the two conditions,

(*i*) If an integer m occurs next to each of the $\phi(m)$ integers $1, a_2, a_3, \cdots, m-1$ prime to and less than m,

(*ii*) then ultimately it must occur next to each of the integers

$$q = jm + 1 \qquad (i = 1, a_2, \cdots, m-1; j = 1, 2, \cdots)$$

as can be seen by simple inductive reasoning. Plainly these numbers comprise all q such that $q > m$, $(q, m) = 1$. Hence *all* relatively prime pairs involving m occur if (*i*) holds for m. But if (*i*) is true for all integers less than m it is true for m, since the $\phi(m)$ integers prime to m are included among $1, 2, \cdots, m-1$ for which (*i*) and (*ii*) both hold. The induction is completed by noting the truth of (*i*) when $m = 2$.

THEOREM 7. *Ultimately, every integer $m > 1$, appears precisely $\phi(m)$ times in a row of the table.*

This is true since m is formed by adding an integer to each of the $\phi(m)$ integers $1, a_2, \cdots, m-1$. (*Cf.* this MONTHLY, 1947, 112, Problem 4236.)

Our concluding remarks return to the configuration of circles which initiated this discussion.

THEOREM 8. *The area-sum of the configuration is*

$$\pi + \frac{\pi\zeta(3)}{\zeta(4)} = 6.6307288,$$

where $\zeta(s) = \sum 1/n^s$ is the Riemann Zeta function.

The area becomes by Theorem 7

$$\lim_{n\to\infty} \sum \frac{\pi}{(A^n)^4} = \pi + \pi \sum_{m=1}^{\infty} \frac{\phi(m)}{m^4}$$

and from p. 249, *loc. cit.*, the result follows.

As might be expected, the same argument shows that the sum of radii diverges.

MATHEMATICAL NOTES

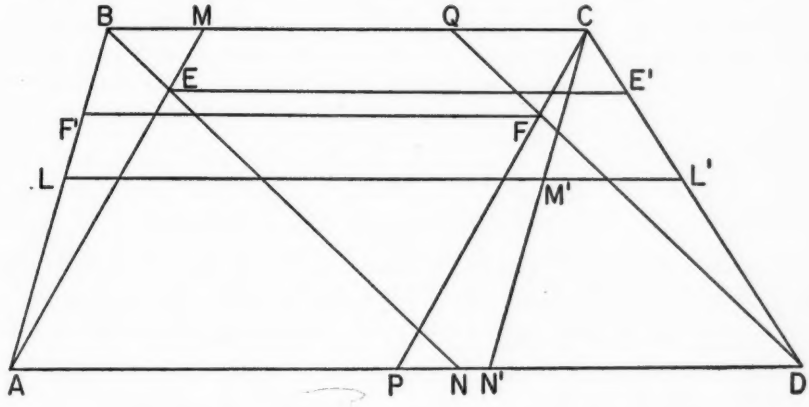
EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

THEOREM ON THE TRAPEZOID

VICTOR THÉBAULT, Tennie, Sarthe, France.

The theorem which we are going to prove is a generalization, believed to be new, of a well known property in the geometry of the trapezoid.



LEMMA. If L *and* L' *divide the non-parallel sides* AB *and* CD *of a trapezoid in the same ratio*

$$BL/BA = k = CL'/CD,$$

then

$$LL' = (1 - k)BC + kAD.$$

Proof. If the parallel to AB through the vertex C cuts LL' and AD at M' and N', by similar triangles as shown in the figure, we have

$$k = CL'/CD = M'L'/N'D = M'L'/(AD - AN') = M'L'/(AD - BC);$$

hence

$$M'L' = k(AD - BC)$$

and

$$LL' = LM' + M'L' = BC + M'L' = (1 - k)BC + kAD.$$

THEOREM. *If through the vertices* A *and* C, B *and* D, *of a trapezoid* $ABCD$, *pairs of parallels are drawn meeting at* E *and* F (E *is the intersection of the lines*

drawn from A and B, F that of the lines drawn from C and D), the distances of E and F to CD and AB, measured along parallels to the parallel sides of the trapezoid, are equal.

Proof. The parallels drawn from A and C meet BC and AD at M and P, those drawn from B and D meet AD and BC at N and Q, and these parallels have any directions. See the figure.

Let EE' and FF' be the distances of the intersections E and F of AM and BN, CP and DQ to CD and AB, measured along parallels to the parallel sides of the trapezoid.

If

$$BC = a, \qquad AD = b, \qquad AP = MC = m, \qquad BQ = ND = n,$$

then

$$k \equiv ME/MA = (a - m)/(a + b - m - n),$$
$$k' \equiv QF/QD = (a - n)/(a + b - m - n),$$

for

$$ME/EA = BM/AN = (a - m)/(b - n),$$
$$QF/FD = QC/PD = (a - n)/(b - m).$$

If we now apply the lemma to the trapezoid $AMCD$, the trapezoid $ABQD$, and the lines EE' and FF' parallel to their parallel sides, we find

$$\begin{aligned} EE' &= (1 - k)MC + kAD = (1 - k)m + kb \\ &= (ab - mn)/(a + b - m - n) \\ &= (1 - k')n + k'b = (1 - k')BQ + k'AD = FF', \end{aligned}$$

and this proves the theorem.

COROLLARY. *In a trapezoid the intersection of the diagonals is the midpoint of the segment between the points where the non-parallel sides meet a line drawn through that intersection and parallel to the parallel sides of the trapezoid.*

This corresponds to the case when $E \equiv F$ is the intersection of the diagonals; then $m = n = 0$ and

$$E'F' = 2ab/(a + b),$$

that is *$E'F'$ is the harmonic mean of the parallel sides of the trapezoid.*

ON A THEOREM OF J. GRIFFITHS*

LUCIEN DROUSSENT, Clermont-Ferrand, France

The English geometer J. Griffiths has established [1] the following theorem:

The Euler circle, circumcircle, orthocentroidal circle, and the orthoptic circle of the Steiner ellipse of a triangle have the orthic axis as common radical axis.

* Translated from the French by Howard Eves.

The following definitions should be recalled.† The *Euler circle* is the nine-point circle; the *orthocentroidal circle* is the circle having the join of the orthocenter and the centroid as diameter; the *Steiner ellipse* is the inscribed ellipse which touches the sides of the triangle at their midpoints; the *orthoptic circle* of an ellipse is the locus of points of intersection of perpendicular tangents to the ellipse, and is frequently referred to as the director circle or the Monge circle of the ellipse; the *orthic axis* is the axis of perspectivity of the given triangle and its orthic triangle.

With the aid of this theorem we shall obtain some new properties of cyclic quadrangles. Consider a quadrangle $ABCD$ inscribed in a circle of center O. Now the Euler circles (ω_a), (ω_b), (ω_c), (ω_d) of the four triangles BCD, CDA, DAB, ABC pass through a point P, called the *anticenter* of the quadrangle, and P is also the point of concurrence of the Wallace lines of the vertices A, B, C, D with the respect to the same triangles [2]. If H_a, H_b, H_c, H_d are the orthocenters of the four triangles, then the altitudes DH_a, DH_b, DH_c of the triangles DBC, DCA, DAB intersect the sides BC, CA, AB of triangle ABC in three collinear points on the Wallace line of D with respect to this triangle. This line passes through the midpoint of the segment DH_d, and there meets the Euler circle (ω_d) of triangle ABC [3]. Since it is clear that this point must be the anticenter P, we have refound the following known property.

The orthocenters of the four triangles formed by the vertices of a cyclic quadrangle are the symmetrics of the vertices with respect to the anticenter P [4].

The lines OH_a, OH_b, OH_c, OH_d are the Euler lines of the four triangles BCD, CDA, DAB, ABC. If G_a, G_b, G_c, G_d are the centroids, and ω_a, ω_b, ω_c, ω_d the centers of the Euler circles of these four triangles, the quadrangles $G_aG_bG_cG_d$ and $\omega_a\omega_b\omega_c\omega_d$ are homothetic with respect to O to the quadrangle $ABCD$, the homothetic ratios being $1/3$ and $1/2$ respectively. The lines H_bH_c, G_bG_c, $\omega_b\omega_c$, BC are then all parallel, and if α is the midpoint of AD, the line $P\alpha$, being the radical axis of (ω_b) and (ω_c), is perpendicular to these four lines.

Now, by Griffiths' Theorem, the orthocentroidal circles (G_bH_b) and (G_cH_c), and the circumcircle (O), have for radical center the point of intersection of the orthic axes of the two triangles CDA and DAB, and this point is also the radical center of the two Euler circles (ω_b) and (ω_c), and the circle (O). This point, then, lies on the radical axis of (G_bH_b) and (G_cH_c) and also on the radical axis of (ω_b) and (ω_c). These two radical axes, each being perpendicular to BC, must therefore coincide. The radical axis of (G_bH_b) and (G_cH_c) is thus the line $P\alpha$, whence we have the following result.

The six radical axes of the orthocentroidal circles of the four triangles formed by the vertices of a cyclic quadrangle are the lines joining the anticenter of the quadrangle to the midpoints of the six sides of the quadrangle. These six circles thus have

† Inserted by the translator.

the anticenter of the quadrangle as a common radical center and also as the center of a common orthogonal circle.

Since the circumcircle and the Euler circle of a triangle are inverse with regard to the polar circle of the triangle [5], it follows that these three circles have a common radical axis, and Griffiths' Theorem could equally well have also included the polar circle of the triangle. By exactly the same argument as above, we may, then, in the last theorem replace "orthocentroidal circles" by either "polar circles" or "orthoptic circles of the Steiner ellipses."

The polar circle of a triangle is real only if the triangle is obtuse, and it is easy to see that either two or four of the polar circles of the four triangles formed by the vertices of a cyclic quadrangle must be real.

References

1. J. Griffiths, Nouvelles Annales de Mathématiques, 1864, p. 345, and 1865, p. 322.
2. See, *e.g.*, Emile Lemoine, Nouvelles Annales de Mathématiques, 1869, p. 47, or R. A. Johnson, *Modern Geometry*, p. 251 ff. (H.E.)
3. R. A. Johnson, art. 327. (H.E.)
4. R. A. Johnson, art. 265. (H.E.).
5. R. A. Johnson, art. 278. (H.E.)

CLASSROOM NOTES

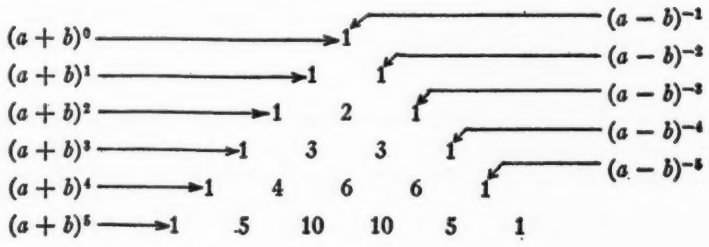
EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania.

PASCAL'S TRIANGLE AND NEGATIVE EXPONENTS

L. V. ROBINSON, University of South Carolina

A most interesting and useful fact apparently overlooked heretofore in the use of Pascal's triangle is that its methods are equally valid for negative, as well as positive, integral exponents. To illustrate this extension in evaluating the binomial coefficients for all integral values of the exponent, it is sufficient to refer to the following scheme:



Thus the horizontal numbers are the coefficients for $(a+b)^n$ and the diagonal ones are for $(a-b)^{-n}$. For example, the binomial expansion of $(a-b)^{-2}$ is, accordingly,

$$(a-b)^{-2} = a^{-2} + 2a^{-3}b + 3a^{-4}b^2 + 4a^{-5}b^3 + 5a^{-6}b^4 + \cdots, \qquad b < a.$$

It seems never inappropriate to point out also to students the use of Pascal's triangle in evaluating the probability of a given number of heads, say, from an equal, or greater, number of tosses of a coin.

UNIFORM CONVERGENCE AND CONTINUITY

L. C. GREEN, Haverford College

Undergraduate students often have difficulty with the concept of uniform convergence and the content of the theorems which arise in connection with it. In particular the relation of uniform convergence to continuity is not always clear to the beginner. This relation may be considered as embodied in the following:

THEOREM. *The limit function $L(x)$ of a sequence or series of continuous functions which converge uniformly in an interval is itself a continuous function in that interval.*

The following simple sequences and the series of which they are the partial sums have been found helpful to the student in understanding the content of the theorem above.

		Uniform Convergence	Non-uniform Convergence
$L(x)$ Continuous	Terms of sequence continuous	Case treated by the theorem	$f_n(x)=\sin nx \quad 0\leqq x\leqq \frac{\pi}{n}$ $f_n(x)=0 \quad \frac{\pi}{n}<x$
	Terms of sequence discontinuous	$f_n(x)=x^n \quad \lvert x\rvert <\frac{1}{2}$ $f_n(x)=2x^n \quad \frac{1}{2}\leqq \lvert x\rvert \leqq a<1$	$f_n(x)=\sin nx \quad 0\leqq x\leqq \frac{\pi}{2n}$ $f_n(x)=0 \quad \frac{\pi}{2n}<x$
$L(x)$ Discontinuous	Terms of sequence continuous	An example here would contradict the theorem	$f_n(x)=(\cos^2 x)^n$
	Terms of sequence discontinuous	$f_n(x)=\frac{1-x^n}{1-x} \quad \lvert x\rvert <\frac{1}{2}$ $f_n(x)=2\,\frac{1-x^n}{1-x} \quad \frac{1}{2}\leqq \lvert x\rvert \leqq a<1$	$f_n(x)=e^{1/x}\frac{1-x^n}{1-x} \quad 0\neq \lvert x\rvert \leqq a<1$ $f_n(0)=1$

INTEGRATION BY PARTS

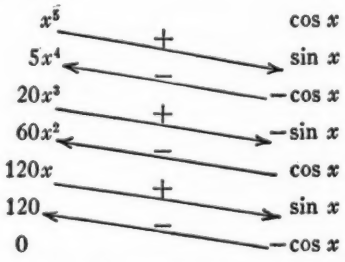
K. W. FOLLEY, Wayne University

A schematic method for integrating by parts the product of two functions of x is illustrated in the following examples.

EXAMPLE 1. The product of a power of x by a function which can be integrated successively.

$$I_1 = \int x^5 \cos x dx.$$

Let $f = x^5$ and $g = \cos x$. Write the derivatives of f with respect to x in one column and the integrals of g with respect to x in a second column, continuing to the row in which $f^{(n)} = 0$.

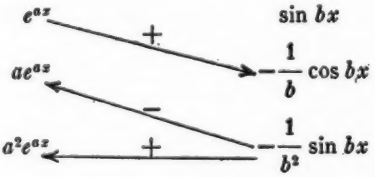


$$I_1 = x^5 \sin x + 5x^4 \cos x - 20x^3 \sin x - 60x^2 \cos x + 120x \sin x + 120 \cos x + C.$$

EXAMPLE 2. The product of an exponential function by a trigonometric function.

$$I_2 = \int e^{ax} \sin bx dx.$$

Let $f = e^{ax}$ and $g = \sin bx$. Proceed as in Example 1, but continue to the row in which the product $f^{(n)}g_n$ is a constant multiple of I_2.

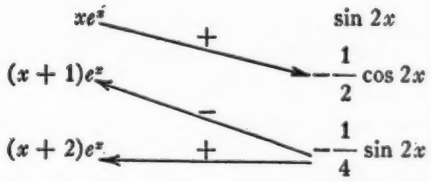


$$I_2 = e^{ax}\left(-\frac{1}{b}\cos bx + \frac{a}{b^2}\sin bx\right) - \frac{a^2}{b^2} I_2$$

$$I_2 = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C.$$

EXAMPLE 3.

$$I_3 = \int xe^x \sin 2x dx.$$



$$I_3 = -\frac{1}{2}xe^x \cos 2x + \frac{1}{4}e^x(x+1)\sin 2x - \frac{1}{4}I_3 - \frac{1}{2}\int e^x \sin 2x dx.$$

The final integral is a special case of Example 2. If we substitute its value and solve for I_3, we obtain

$$I_3 = \frac{e^x}{25}[(3+5x)\sin 2x + (4-10x)\cos 2x] + C.$$

The justification for the preceding method follows from the continued application of the formula for integration by parts.

$$\int fg dx = fg_1 - \int f'g_1 dx = fg_1 - f'g_2 + \int f''g_2 dx = \cdots$$

$$= fg_1 - f'g_2 + \cdots + (-1)^{n-1}f^{(n-1)}g_n + (-1)^n \int f^{(n)}g_n dx,$$

where

$$f^{(i+1)} = \frac{d}{dx}f^{(i)} \quad \text{and} \quad g_{i+1} = \int g_i dx.$$

A SIMPLIFICATION OF THE SECOND DERIVATIVE TEST

L. S. LAWS, University of Minnesota

This note concerns a simplification of the second derivative test for maximum-minimum in the case where $y=\phi(x)/f(x)$. Obtain $y'=N(x)/[f(x)]^2$ by the usual method for differentiating a quotient of functions, and find a value x_1 such that $N(x_1)=0$ and $f(x_1)\neq 0$.

To prove: $N'(x_1)$ has the same sign as $y''(x_1)$.

Proof:

$$y'' = \frac{[f(x)]^2 N'(x) - N(x)[2f(x)f'(x)]}{[f(x)]^4},$$

but since $f(x_1)\neq 0$ and $N(x_1)=0$, the quantity $[f(x_1)]^2$ is positive and $N'(x_1)$ will

have the same sign as $y''(x_1)$. Therefore, y will have a relative maximum at $x=x_1$ if $y'(x_1)=0$ and $N'(x_1)<0$; or y will have a relative minimum at $x=x_1$ if $y'(x_1)=0$ while $N'(x_1)>0$.

VECTOR DERIVATION OF THE SINE AND COSINE LAWS IN SPHERICAL TRIGONOMETRY

J. F. HEYDA, Indianapolis, Indiana

The usual derivation of these formulas from a study of the relationships involved in the proper plane triangles frequently leaves the student in a three-dimensional fog, part of this fog arising as a result of the instructor's poor diagrams. The present derivation requires little artistic ability on the part of the instructor, providing that he has spent a session or two explaining the fundamental operations with vectors including dot and cross multiplication.

Assume an arbitrary spherical triangle with vertices at A, B, C with opposite sides a, b, c and let $\mathbf{e}_A$, $\mathbf{e}_B$, $\mathbf{e}_C$ denote unit vectors directed from the center of the sphere to the vertices A, B, C, respectively, Also let $\mathbf{t}_{AB}$ denote the unit tangent vector, tangent to arc AB at A and directed toward B, with equivalent interpretations for $\mathbf{t}_{AB}$, $\mathbf{t}_{BC}$, *etc.* The unit normals

$$\mathbf{n}_{AB} = \mathbf{e}_A \times \mathbf{t}_{AB}, \qquad \mathbf{n}_{BC} = \mathbf{e}_B \times \mathbf{t}_{BC}, \qquad \mathbf{n}_{AC} = \mathbf{e}_C \times \mathbf{t}_{CA}$$

to the planes of arcs AB, BC, AC, respectively, are then directed toward the interior of the solid angle O-ABC, our sphere in question having unit radius.

To prove the cosine law we note first that

$$\mathbf{e}_C = \cos b\, \mathbf{e}_A + \sin b\, \mathbf{t}_{AC}$$

and also that

$$\mathbf{t}_{AC} = \cos A\, \mathbf{t}_{AB} + \sin A\, \mathbf{n}_{AB},$$

whence we have

$$\mathbf{e}_C = \cos b\, \mathbf{e}_A + \sin b \cos A\, \mathbf{t}_{AB} + \sin b \sin A\, \mathbf{n}_{AB}.$$

Dotting both sides of the latter equation with $\mathbf{e}_B$, we obtain

$$\mathbf{e}_B \cdot \mathbf{e}_C = \cos a = \cos b \cos c + \sin b \sin c \cos A$$

as desired.

The proof of the sine law follows immediately upon dotting both sides of the identity

$$\mathbf{e}_C = \cos b\, \mathbf{e}_A + \sin b\, \mathbf{t}_{AC} = \cos a\, \mathbf{e}_B + \sin a\, \mathbf{t}_{BC}$$

with $\mathbf{n}_{AB}$, observing that

$$\mathbf{n}_{AB} \cdot \mathbf{e}_A = \mathbf{n}_{AB} \cdot \mathbf{e}_B = 0,$$

so that the desired result is

$$\sin b \sin A = \sin a \sin B.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 791. *Proposed by G. W. Walker, Buffalo, N.Y.*

The court mathematician once received his salary for a year's service all at one time, and all in silver "dollars," which he proceeded to arrange in nine unequal piles, making a magic square. The king looked, and admired, but complained that there was not a single prime number in any of the piles. "If I had but nine coins more," said the mathematician, "I could add one coin to each pile and make a magic square with every number prime." They investigated, and found that this was indeed true. The king was about to give him nine "dollars" more, when the court jester said, "Wait!." Then the jester subtracted one coin from each pile instead; and they found in this case also a magic square with every element a prime number. The jester kept the nine "dollars." How much salary must the mathematician have been receiving?

E 792. *Proposed by N. S. Mendelsohn, University of Manitoba*

Show that a pack of $2n$ cards is brought back to its original position in at most $2n-2$ perfect riffle shuffles. (A perfect riffle shuffle is one which sends the cards from the arrangement $1, 2, 3, 4, \cdots, 2n-1, 2n$ to the arrangement $1, n+1, 2, n+2, \cdots, n, 2n$.)

E 793. *Proposed by Joseph Rosenbaum, The Milford School, Connecticut*

With straight edge alone construct a hexagon which can possess both an inscribed and a circumscribed conic.

E 794. *Proposed by Huan-Ting Kuo, National Wuhan University, China*

Show that

$$\begin{vmatrix} \binom{r+1}{r} & \binom{r+1}{r+1} & 0 & 0 & \cdots & 0 \\ \binom{r+2}{r} & \binom{r+2}{r+1} & \binom{r+2}{r+2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{r} & \binom{n}{r+1} & \binom{n}{r+2} & \binom{n}{r+3} & \cdots & \binom{n}{n-1} \end{vmatrix} = \binom{n}{r}.$$

E 795. *Proposed by N. A. Court, University of Oklahoma*

The pairs of points U', U''; V', V''; W', W'' are marked, respectively, on the edges DA, DB, DC of a tetrahedron $ABCD$. Determine three points U, V, W on the edges BC, CA, AB, respectively, so that the three lines joining the vertices of each of the triangles $UV'W''$, $VW'U''$, $WU'V''$, UVW to the corresponding vertices of the triangles DCB, DAC, DBA, ABC, respectively, shall have a point in common.

SOLUTIONS

A Problem in Stability

E 761 [1947, 163]. *Proposed by C. R. Perisho, Nebraska Wesleyan University.*

An object with a smooth lower plane surface, and center of gravity h units above this surface, is balanced on a sphere of radius R. Find the relation between h and R which insures stability of the object under small displacements.

I. *Solution by E. S. Keeping, University of Alberta.* If the object of mass M rocks through a small angle θ, its gravitational potential energy, relative to the center of the sphere, is

$$V = Mg[R\theta \sin\theta + (R + h)\cos\theta],$$

provided that no slipping occurs. Therefore

$$\begin{aligned} dV/d\theta &= Mg(R\theta\cos\theta - h\sin\theta) \\ &= 0 \quad \text{at} \quad \theta = 0, \end{aligned}$$

so that equilibrium exists. Also

$$\begin{aligned} d^2V/d\theta^2 &= Mg[(R - h)\cos\theta - R\theta\sin\theta] \\ &= Mg(R - h) \quad \text{at} \quad \theta = 0. \end{aligned}$$

Hence the equilibrium is stable if $h<R$, unstable if $h>R$.

If $h=R$, then $d^3V/d\theta^3=0$ and $d^4V/d\theta^4=-2Mgh$ at $\theta=0$, so that equilibrium is unstable in this case.

The problem states that the object has a "smooth" under surface. If this means that there is no friction whatever, the equilibrium is bound to be unstable, since the slightest disturbance will cause the object to slide off.

If there is to be a restoring force at angle θ, it is necessary that $\tan\theta \leqq \mu$; and hence the permissible displacement (for $h<R$) will vanish with μ.

II. *Solution by R. A. Bradley, University of North Carolina.* Let N be the point on the sphere vertically above its center O, C the point of contact of the object when rocked through a small angle $NOC=\theta$, A the position of the center of gravity of the object when in its displaced position, and B the foot of the perpendicular from A onto the tangent plane at C. For stable equilibrium A must lie between ON and the vertical line through C. That is, we must have

angle BAC larger than θ, whence

$$\tan BAC = BC/BA = R\theta/h > \tan\theta > \theta.$$

This implies that $h<R$. The equilibrium is unstable if $h \geqq R$.

Also solved by L. M. Kelly, J. C. Miller, G. W. Walker, and the proposer.

Rufus Crane pointed out that this problem occurs as problem 275 on page 242 of *Engineering Mechanics* by Timoshenko and Young. The problem is there solved by virtual displacements with the result given above.

The Arbelos

E 762 [1947, 163]. *Proposed by J. R. Van Andel, Naval Air Experimental Station, Philadelphia, Pa.*

Let A_1 and A_2 be two circles with radii a_1 and a_2 and centers $(a_1, 0)$ and $(a_2, 0)$, respectively, with $a_2>a_1>0$. Let C be any circle in the crescent shaped area M between A_1 and A_2, and tangent to both A_1 and A_2.

(a) The locus of the center of C as it sweeps out M is an ellipse with semi-axes $(a_1+a_2)/2$ and $\sqrt{a_1a_2}$.

(b) If C_t is a circle of radius r_t and center $P_t(x_t, y_t)$ where

$$\begin{aligned} r_t &= a_1a_2(a_2 - a_1)\phi_t, \\ x_t &= a_1a_2(a_2 + a_1)\phi_t, \\ y_t &= 2tr_t \\ \phi_t^{-1} &= a_1a_2 + t^2(a_2 - a_1)^2, \end{aligned}$$

then, for any real value of t, C_t lies in M and is tangent to A_1, A_2, and C_{t-1}.

Solution by Norman Anning, Ann Arbor, Michigan. (a) This part is elementary. A glance at a figure shows that the center of C is always in such a position that the sum of its distances from $(a_1, 0)$ and $(a_2, 0)$ is a_1+a_2. The rest follows.

(b) If (X, Y) is a point on C_t, then

$$(1) \qquad (X^2 + Y^2)\phi_t^{-1} - 2a_1a_2(a_1 + a_2)X - 4ta_1a_2(a_2 - a_1)Y + 4a_1^2a_2^2 = 0.$$

Apply to this circle the inversion

$$X = 4a_1a_2x/(x^2 + y^2), \qquad Y = 4a_1a_2y/(x^2 + y^2).$$

Then (1) becomes

$$x^2 + y^2 - 2(a_1 + a_2)x - 4t(a_2 - a_1)y + 4\phi_t^{-1} = 0,$$

which may be written as

$$(2) \qquad (x - a_1 - a_2)^2 + (y - 2ta_2 + 2ta_1)^2 = (a_2 - a_1)^2.$$

With t as parameter, (2) is the family of equal circles which touch the

parallel lines $x=2a_1$ and $x=2a_2$. In this family, for every t, the circle C_t is tangent to C_{t-1} because the distance between their centers is equal to the diameter of either.

Now invert again. Circle C_0 inverts into itself and (2) inverts into (1). The line $x=2a_2$ inverts into the circle A_1, the inner boundary of the arbelos. Similarly, $x=2a_1$ inverts into A_2, the outer boundary. Since it is well known that inversion turns circles into circles and preserves contacts, the proof of the stated theorem is complete.

One tracing the history of the problem would find it under *arbelos*. See R. Johnson's *Modern Geometry*, for instance. The neatest of the properties, $y_t=2tr_t$, appears in book 4 of Pappus's *Collection*. See Ivor Thomas, *Greek Mathematical Works*, II (Loeb Classical Library, No. 362), p. 578.

Also solved by Paul Brock, Rufus Crane, G. A. Williams, and the proposer.

The proposer pointed out that J. Steiner in *Geometrische Betrachtungen* (1826) discussed, in particular, the chains of circles corresponding to the sequences $t=0,\ 1,\ 2,\ \cdots$ and $t=1/2,\ 3/2,\ 5/2,\ \cdots$. Williams mentioned several additional properties of the figure which easily follow from the inversion. For instance, the line joining the points of contact of C_t with A_1 and A_2 passes through the fixed point $(2a_1a_2/(a_1+a_2),\ 0)$; the common internal tangent of C_t and C_{t-1} passes through this same point; the four points consisting of the origin, the centers of A_1 and A_2, and the above point form a harmonic set.

If the diameter of A_1 is taken as two-thirds that of A_2, then r_1 is one-seventh the diameter of A_2. Of this particular figure Victor Thébault has stated a very pretty property. Let the diameter of A_2 taken along the line of centers of A_1 and A_2 be OB, and let BM be the tangent from B to the circle A_1. Then the circle on BM as diameter is tangent to the circle C_1.

Editorial Note. In connection with E 762 see *On a Generalization of the Arbelos*, by M. G. Gaba, this MONTHLY [1940, 19].

A New Property of the Monge Point

E 763 [1947, 163]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The lines joining the orthocenters of the faces of tetrahedron to the reflections in these faces of the points of intersection of the corresponding altitudes with the circumsphere, are concurrent at the Monge point of the tetrahedron.

Solution by L. M. Kelly, University of Missouri. In the tetrahedron $ABCD$ let M, G, O be the Monge point, centroid, and circumcenter, and let H_a, G_a, O_a be the orthocenter, centroid, and circumcenter of the face BCD. Denote by A', E, and G' the feet of the perpendiculars from A, M, and G on the face BCD, and let the altitude AA' cut the circumsphere at X. (A figure, with the same lettering, may be found on p. 69 of N. A. Court's *Modern Pure Solid Geometry*.) For convenience let us set $AA'=h$, $A'X=x$, $OO_a=p$. Then we have

$$h/4 = GG' = (ME + p)/2,$$

whence

$$ME = h/2 - p.$$

Again

$$2(p + x) = h + x,$$

or

$$x = h - 2p.$$

Thus, since E is the midpoint of $A'H_a$, the line H_aM will pass through the reflection of X in A'. The theorem now follows.

Also solved by the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4270. *Proposed by S. H. Gould, Victoria College, Toronto*

Let b be a fixed positive integer, $m = 1, 2, \cdots$, and q irrational with $0 < q < 1$. Call the interval $(m, m+1)$ a gap if it does not contain a multiple of $b+q$. Prove that every set of b successive gaps contains exactly one multiple of $1+b/q$.

This is a generalization of problem 3173, [1927, 159].

4271. *Proposed by N. A. Court, University of Oklahoma*

The external bisectors of the three face angles of each trihedron of a given tetrahedron are coplanar. The four planes form a second tetrahedron. Show that the lines joining corresponding vertices of the two tetrahedrons form, in general, a hyperbolic group.

4272. *Proposed by A. L. Epstein, Asbury Park, N. J.*

Given the sequence n_i where $n_1 = a$, $n_2 = b$, and $n_{i+2} = n_{i+1} + n_i$, $i = 1, 2, \cdots$. Show that

$$\sum_{i=1}^{4k-2} n_i = r_k n_{2k+1}$$

and determine the form of r_k. (This is suggested by the familiar trick in which the subject selects n_1, n_2, and computes n_3, n_4, $\cdots$, n_7. He tells the operator the result n_7, and continues the calculation through n_{10}. When he adds the ten numbers he finds the operator has anticipated his result, which must be $11n_7$.)

4273. *Proposed by I. S. Cohen, University of Pennsylvania*

Prove that for any positive odd integer n, $\cos\theta$ and $\sin\theta$ are rational functions of $\cos^n\theta$ and $\sin^n\theta$ with rational coefficients. Find the explicit expressions in the case $n=3$.

4274. *Proposed by R. Bouvaist, Vincelles, Saône-et-Loire, France*

Let A, B, C, D be arbitrary points on an equilateral hyperbola (H), and let A', B', C', D' be the corresponding diameterically opposite points. (1) The isogonal conjugates of A', B', C', D' with respect to the triangles BCD, CDA, DAB, ABC, respectively, coincide in the same point P. (2) The isogonal conjugates of A, B, C, D with respect to the triangles $B'C'D'$, $C'D'A'$, $D'A'B'$, $A'B'C'$, respectively, coincide in the same point P'. (3) P and P' are diametrically opposite on (H).

SOLUTIONS

Special Pythagorean Triangles

4205 [1946, 278]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a right triangle has sides of integral lengths and the sum of the sides forming the right angle is a square, then the sum of the cubes of these two sides is the sum of two squares. Can the hypothenuse be a square?

Solution by Paul Bateman, Philadelphia, Pennsylvania. If a triple of positive integers $(a, b; c)$ satisfies the equation $a^2+b^2=c^2$, then either just one of the three numbers is even or else all three are even. Hence $\frac{1}{2}(c-b+a)$ and $\frac{1}{2}(c+b-a)$ are integers. Now

$$\begin{aligned} a^2 - ab + b^2 &= \tfrac{1}{2}[a^2 + b^2 + (a-b)^2] \\ &= [\tfrac{1}{2}(c+b-a)]^2 + [\tfrac{1}{2}(c-b+a)]^2, \end{aligned}$$

and hence a^2-ab+b^2 is always a sum of two squares. Thus $a^3+b^3=(a+b)(a^2-ab+b^2)$ is the sum of two squares if and only if $a+b$ is a square or the sum of two squares.

The hypothenuse can be a square. There are infinitely many such triangles, the smallest of which was given by Fermat:

$$(4\,565\,486\,027\,761,\ 1\,061\,652\,293\,520;\ 4\,687\,298\,610\,289).$$

See Carmichael, *Diophantine Analysis*, p. 77, Ex. 1; Dickson, *History of the Theory of Numbers*, V. 2, pp. 620–627; Uspensky and Heaslet, *Elementary Number Theory*, pp. 413–419.

Solved also by Murray Barber, D. H. Browne, Daniel Finkel, C. D. Olds, and the Proposer.

Four Collinear Orthocenters

4210 [1946, 397]. *Proposed by R. Goormaghtigh, Bruges, Belgium*

If the parallels to the sides of triangle ABC drawn through a point P on the circumcircle meet that circle again at A', B', C', the orthocenter H of ABC and those α, β, γ of $A'BC$, $B'CA$, $C'AB$ are on a straight line perpendicular to the Simson line Δ of P as to ABC; and the center of gravity of α, β, γ divides into the ratio 2:1 the distance from H to the circumdiameter parallel to Δ.

I. *Solution by J. H. Butchart, Arizona State College at Flagstaff.* The vector AH equals the vector $A'\alpha$ since each is twice the vector from the circumcenter O to the side BC and perpendicular to BC. Hence the vector $H\alpha$ equals the vector AA'. If K is the point where the circumcircle meets the perpendicular from P to BC, it is known that AK is parallel to Δ. Since A' and K are ends of a diameter, AA' is perpendicular to AK, and hence $H\alpha$ is perpendicular to Δ. By the same argument $H\beta$ and $H\gamma$ are also perpendicular to Δ, and the first statement of the proposal is proved.

For the second part, note that the diameter parallel to Δ bisects AA', BB' and CC'. Since the vector sum of OA, OB and OC is OH, the projection of this vector sum on $H\alpha$ is DH, where D is the point in which $H\alpha$ meets this diameter. Then if the vectors AA', BB' and CC' are laid off from H, their sum is the vector $2HD$. Hence the center of gravity of α, β, γ is the end of the vector $2HD/3$.

II. *Solution by the Proposer.* Let ABC be the base circle in a system of complex coördinates, $\boldsymbol{P}$ being the unit point. Denote by t_1, t_2, t_3 the coördinates of A, B, C and by s_1, s_2, s_3 their symmetric functions

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_1t_2 + t_2t_3 + t_3t_1, \qquad s_3 = t_1t_2t_3.$$

Then s_1 is the coördinate of H. The equation of Δ is

$$x - s_3\bar{x} = \tfrac{1}{2}(1 + s_1 - s_2 - s_3)$$

and that of the circumdiameter δ parallel to Δ is

$$x - s_3\bar{x} = 0;$$

hence the image H' of s_1 in δ is s_2.

The coördinate of A' being t_2t_3, the point α has as coördinate

$$t_2 + t_3 + t_2t_3,$$

and α is on the line

$$x - s_1 + s_3(\bar{x} - \bar{s}_1) = 0,$$

which contains H and is perpendicular to Δ.

Further, the center of gravity of α, β, γ is $(2s_1+s_3)/3$, the point dividing HH' into the ratio 1:2.

Solved also by J. W. Clawson, H. E. Fettis, and Ou Li.

The Quadrilateral

4214 [1946, 397]. *Proposed by V. Thébault, Tennie, Sarthe, France*

On the sides AB, CD of an arbitrary quadrangle $ABCD$ isosceles triangles are constructed with the same sense $A'AB$, $C'CD$, with the base angle θ; and on the sides BC, DA the isosceles triangles $B'BC$, $D'DA$ with the base angle $\pi/2-\theta$ and in the same sense as the first. Prove that (1) The lines $A'C'$, $B'D'$ are perpendicular and that the lengths of the segments are in the ratio $\tan\theta$. (2) The centroid of the quandrangle is on the straight line joining the midpoints of $A'C'$, $B'D'$ which it divides in the ratio $\tan^2\theta$. (3) Find the locus of the midpoints of the sides and diagonals of the quadrangle $A'B'C'D'$ and the envelope of all these lines.

Solution by R. Goormaghtigh, Bruges, Belgium

(1) Let a, b, c, d be the complex coordinates of A, B, C, D, the origin being O; then, if $\tan\theta$ is denoted by k, the points A', C', B', D' are

A': $[a+b+ik(a-b)]/2$, $\qquad$ C': $[c+d+ik(c-d)]/2$,

B': $[b+c+i(b-c)/k]/2$, $\qquad$ D': $[d+a+i(d-a)/k]/2$.

The segments $A'C'$ and $B'D'$ are equipollent to those having O for origin and the points

$$[c+d-a-b+ik(c-d-a+b)]/2,$$
$$[d+a-b-c+i(d-a-b+c)/k]/2$$

for extremities; since the first expression is equal to the second multiplied by $-ik$, the segments $A'C'$ and $B'D'$ are perpendicular and in the ratio k. (2) The midpoints M and N of $A'C'$ and $B'D'$ are

$$(a+b+c+d)/4+ik(a-b+c-d)/4,$$
$$(a+b+c+d)/4-i(a-b+c-d)/4k.$$

Hence the centroid G of A, B, C, D, the coordinate of which is $(a+b+c+d)/4$ and the points M, N are collinear and MG and GN are in the ratio $k^2:1$.

The same proof applies to the following *generalization*:

On the sides $A_1A_2, A_3A_4, \cdots, A_{2n-1}A_{2n}$ *of a* $2n$*-sided polygon* $A_1A_2\cdots A_{2n}$ *are constructed in the same sense isosceles triangles* $A_1'A_1A_2$, $A_3'A_3A_4, \cdots,$ $A_{2n-1}'A_{2n-1}A_{2n}$ *with the base angle* θ; *on the sides* A_2A_3, $A_4A_5, \cdots, A_{2n}A_1$ *of the same polygon are constructed in the same sense triangles* $A_2'A_2A_3$, $A_4'A_4A_5, \cdots,$ $A_{2n}'A_{2n}A_1$ *with the base angle* ϕ. *Then the centroid* G *of the vertices of the polygon is on the straight line joining the centroids* M *and* N *of the two groups of points* $A_1', A_3', \cdots, A_{2n-1}'$ *and* $A_2', A_4', \cdots, A_{2n}'$ *and* MG *and* GN *are in the ratio* $\tan\theta/\tan\phi$.

(3) The point-sets described by A', B', C', D' on the axes of AB, BC, CD, DA being similar, their joins envelope parabolas tangent to those axes and the midpoints of all the segments between any two of them have as loci straight lines.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Preparatory Business Mathematics. By L. L. Smail. New York, The Ronald Press Co., 1947, 10+240 pages. $2.75.

Briefly this is a good book, though somewhat condensed, with the proper emphasis on the mathematical preparation for business and enough applications in later chapters to sustain interest. The somewhat skeletonized presentation found in the early chapters of the book is justified by the opening sentences: "It is assumed that the reader has had a course in elementary algebra. The first chapter (and a large part of the next few chapters) is intended to give merely a brief review of some of the topics of elementary algebra . . . . "

Teachers generally will approve of the generous use of italicized letters and bold face type to call attention to definitions, fundamental procedures and formulas. The material is broken up into convenient assignment units interspersed with ample sets of exercises, the answers to which are all given. The figures are well drawn with two notable exceptions:

(1) Fig. 11 is intended to show the approximations to the real roots of a cubic, but the curve quite obviously does not pass through any of the three designated points.

(2) Fig. 28, the graph of the parabola is clearly not symmetrical to its axis.

The most serious criticism of the book seems to be the treatment of the number system. Rational, irrational, and imaginary numbers are not defined, and except for one reference to irrational exponents, these words do not appear in the Index.

It may be argued that a knowledge of the classification of the number system is superfluous for business students, hence it is unnecessary to define the various kinds of numbers. In the interest of consistency then all references to rational, irrational, and imaginary numbers should be avoided. This unfortunately has not been done as noted below:

(1) *p.* 50 " . . . if there is no positive or negative nth root, any one of the nth roots may be taken as the principal root."

(2) *p.* 53 "The preceding definitions of powers will not apply if the expression is an irrational number, as for example $3^{\sqrt{2}}$."

(3) *p.* 54 "The irrational number $\sqrt{2}$ may be represented by the unending sequence of rational numbers 1, 1.4, 1.41, 1.414, 1.4142, ... "

(4) *p.* 64 "Every positive number has a logarithm which is a real number; but the logarithm of a negative number is an imaginary or a complex number."

There are good reasons for extending the definitions of the arithmetic and geometric means to include means of more than two numbers, which the author failed to do, and to introduce the concept of weighted means considering their

extensive use in the study of index numbers in statistics. While on the subject of averages, the reviewer still entertains the hope that some elementary textbook writer, especially one who is writing a book of this type, will recognize the statistical implications of writing the formula for the sum of n terms of an arithmetical progression as,

$$S = n\,\frac{a+l}{2}$$

rather than,

$$S = \frac{n}{2}\,(a+l).$$

It is gratifying to have a paragraph on "Standard (or Scientific) Notation for Numbers in Decimal Form" which is a very useful tool as indicated on p. 59. Yet it is surprising that the advantage of this notation is not recognized on p. 69, by giving the following rule for finding the characteristic of the logarithm of N: The characteristic of the logarithm of N is the exponent of 10 used when N is expressed in the scientific notation.

Since the idea of the minimum value of a function is so fundamental in the study of statistics the author did well to include a determination of the minimum value of a quadratic function as given in Article 134. However it would have been preferable, pedagogically, to have used the same procedure in completing the square as was used earlier in Article 42 and have written,

$$y = K + \frac{1}{A}\,(Ax+B)^2,$$

from which both of the theorems on maximum and minimum values of y would have followed immediately and the rôle of A have been more clearly and readily appreciated.

The author is to be commended for attempting to give business students a more adequate understanding of analytical geometry, especially curve fitting by the method of least squares, than is generally done in books of this type, and for stressing the uses of logarithmic graph paper. Some doubts exist however as to the advisability of giving such a casual introduction of the concepts of symmetry and asymptotes as is done here.

It is quite generally accepted that the American Experience Table was based on mortality statistics deduced from the experience of the Mutual Life Insurance Company of New York rather than "from the accumulated results of life insurance companies" as stated on p. 206.

The three tables:

I Logarithms of Numbers
II Squares and Square Roots of Numbers
III American Experience Table of Mortality

are adequate for the purposes of the author.

The book is a worthy addition to published books of this type and will be found quite teachable for students with a good background in elementary algebra.

F. S. HARPER

Mathematics as a Culture Clue, and Other Essays. (Vol. I of *The Collected Works.*) By C. J. Keyser. New York, Scripta Mathematica, 1947. 7+277 pages. $3.75.

In these essays, which are "designed primarily for laymen" (*see e.g., p.* 25) the author makes an energetic attempt to explain the important mathematical concept of a system of postulates, a concept whose range of possible applicarions is very wide, embracing (*p.* 19) "all subject-matters, material or mental, physical or psychical, organic or inorganic."

Part of the reason for expounding the subject so fully is that Hilbert in his *Foundations of Geometry* was guilty, according to Keyser, (*cf. pp.* 15, 107, 115), of "an obfuscating and misleading practice." In setting forth his system of axioms and their implications

> Hilbert at the same time applies it to geometric subject-matter—a fact shown by his use of certain geometric figures. Because the reader's attention is thus concentrated upon the rising structure of the geometric doctrine he perceives but dimly or not at all the simultaneously rising structure of the doctrinal function, of which the geometric doctrine is but one of countless values. And his false impression is, unfortunately, confirmed by Hilbert's calling his book *The Foundations of Geometry*, a flagrant misnomer since it has, essentially, no more to do with geometry than with a thousand other matters—a fact exhibited with ample detail in my *Mathematical Philosophy.* . . . It is not very difficult to show, and in my *Mathematical Philosophy* I have shown, that the proper values are of different types, some of them geometric, some of them arithmetical or numerical, and some of them neither the one nor the other.

But even when it was thus amply pointed out to him that his points and lines could be interpreted in other ways, Hilbert did not change the title of his book nor remove its diagrams. He may have felt that the title might stand provisionally until new applications should attain the same importance as the old ones, and as for the diagrams, they made the text easier to follow, an advantage which could be weighed against their obfuscating tendencies and perhaps be regarded as a compensation for the reader of dim perception.

Seven of Keyser's twelve essays are essentially book reviews, of works by such writers as Peirce, Havelock Ellis, Pareto, Spengler and Korzybski. Of Spengler's book *The Decline of the West* he says (*p.* 47) "it is, with perhaps a single exception, the most mind-stirring work of our time"; and of Korzybski's *Science and Sanity* (*p.* 153) "this work, taken as a whole, is beyond comparison the most momentous single contribution that has ever been made to our knowledge and understanding of what is essential and distinctive in the nature of Man."

It is the essay on Spengler's book which provides the title *Mathematics as a Culture Clue.* This essay has nothing to do with mathematics as part of the cul-

ture of an individual, but as Keyser explains (*p.* 45), "my purpose is to submit a certain grave thesis" of Spengler's, which is (*p.* 52) that "each of the expression-forms (mathematics, music, architecture, *etc.*) of a given culture (Old Egyptian, Babylonian, Chinese, etc., *cf.* p. 43) essentially bears the image of each of its fellow forms . . . revealing in and of itself what is distinctive in that Culture's soul."

In illustration of Spengler's thesis, Keyser writes (*p.* 52)

> and so, were a Culture completely lost save for a single one of its expression forms, it would be theoretically possible, and in a measure practically possible, to determine from that sole survivor what the lost forms and the total Culture itself were essentially like, much as, since Cuvier, it has been possible to determine from a single fossil bone of an extinct animal, not only its family, genus, and species, but even the external form of the individual "with certainty and precision."

The project is an interesting one, with many applications; but there seems to have been some delay in carrying them out.

One of the essays, with the title *Three Great Synonyms: Relation, Transformation, Function,* is slightly more mathematical than the others. Its painstaking explanations are somewhat marred by printer's lapses whereby, for example, the three words which in the beginning are synonymous and possess synonymy (*p.* 218) become in the end *synonomous* and possess *synonomy* (*p.* 234). The familiar linguistic principle of dissimilation makes these errors far harder for the proof-reader to detect than a misspelling like *anonomous.* The same principle is at work in the phrases "Human Curosity" (*p.* 263), "irratating slovenliness" (*p.* 274) *etc.* Misprints in general are not rare throughout the book, *e.g.* "summitless heirarchy" (*p.* 28), "our outreachings for Diety" (*p.* 42) *etc.* Assimilation has been at work, too, on the word 'deduction,' for example, (*pp.* 80, 84): "had Gauss known . . . that mathematics and natural science are separated by a chasm as deep and unbridgeable as that which sunders Logical Deducation from Experimental Observation. . . . "

In the course of the book the lay reader is made acquainted with various mathematical concepts, *e.g.* the notion of *Limit* (*p.* 25), "the subtlety of whose meaning it would not be easy to exaggerate," the "essential" point being (*pp.* 26, 27) that "the variable never reaches its limit." The task of the popularizer is hard (Poincaré "not only failed but failed conspicuously," *p.* 72), so no one will object to this definition, seeing that it is used to "disclose in perfect light" (*pp.* 25, 28)—thereby correcting Plato and many others (*pp.* 24, 27)—the ideals of justice, freedom, beauty, happiness, wisdom, moral good, power, clarity, skill, piety, *etc.* (*p.* 27).

But the purpose of the book as a whole is to make clear the essential nature of all mathematics, which is summed up in many places. Thus (*p.* 113), if we let P denote the set of postulates and T a theorem, then the mathematician will say: "I have not . . . proved P to be true, nor have I proved T to be true; what I have proved is the proposition, P implies T." In their preface, the publishing committee says:

> when the mind of a scholar, philosopher or just a cultured layman gets confused and be-

wildered by the great complexity of scientific data or of political events he takes out a volume—
(P) any volume— of Keyser's and gets his mind geared to the rhythm of the cleansing, crystal-clear current flowing in every one of Professor Keyser's works.

and a few lines later:

(T) they believe that by making the Collected Works available they . . . are rendering a genuine service to the cause of clear thinking.

Let us denote their first statement by P and what they believe by T, and give hearty assent to the proposition, P implies T.

S. H. GOULD

NEW BOOKS RECEIVED

Applied Industrial Mathematics. By O. B. Jones. New York, Prentice-Hall, Inc., 1947. 8+342 pages. $3.00.

Intermediate Algebra for Colleges. By E. B. Miller. New York, Ronald Press, 1947. 10+361 pages. $2.50.

The Elements of Aerofoil and Airscrew Theory. Second Edition. By H. Glauert. Cambridge, at the University Press. New York, The Macmillan Company, 1947. 226 pages. $4.00.

Calculus. Revised Edition. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1947. 12+568 pages. $3.75.

Money, Credit and Finance. Revised Edition. By G. F. Luthringer, L. V. Chandler and D. C. Cline. Boston, D. C. Heath and Co., 1947. 7+389 pages. $2.75.

Natural Philosophy of the Science of Physics: Chemistry and Engineering. By C. A. P. Turner. Columbus, Ohio, 1947. 31+320 pages. $6.00.

Introduction Mathématique aux Théories Physiques Modernes, Part I. by M. Morand. Paris, Vuibert, 1947. 140 pages. 350 Fr.

Tables of the Bessel Functions of the First Kind of Orders Four, Five, and Six. (Annals of the Computation Laboratory of Harvard University, Vol. 5.) By the Staff of the Computation Laboratory, Harvard University. Cambridge, Harvard University Press, 1947. $10.00.

Tables of the Bessel Functions of the First Kind of Orders Seven, Eight, and Nine. (Annals of the Computation Laboratory of Harvard University, Vol. 6.) By the Staff of the Computation Laboratory, Harvard University Press, 1947. $10.00.

Theory of Games and Economic Behavior. Second Edition. By J. von Neumann and G. Morgenstern. Princeton University Press, 1947. 18+641 pages. $10.00.

Time, Knowledge, and the Nebulae. By M. Johnson. New York, Dover Publications, 1947. 189 pages. $2.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS, 1946–47

Pi Mu Epsilon, University of Kentucky

The *Kentucky Alpha* Chapter of *Pi Mu Epsilon* held six regular meetings. Faculty members of the mathematics department and graduate students in the department presented the following discussions:

Pursuit curves, by Dr. L. W. Cohen

Boolean algebra, by Dr. A. A. Grau

A problem in probability, by Dr. Casper Goffman

Certain groups, by Donald Rose.

The *Kentucky Alpha* Chapter sponsors the *White Mathematics Club*. This year a cash prize was presented to an undergraduate student for outstanding work.

There were two initiations during the year at which twenty-one new members were received into the chapter.

The officers for 1946–47 were: Director, Dr. H. H. Downing; Vice-Director, Virginia Rohde; Secretary-Treasurer, Louise Knifley; Librarian, Dr. C. G. Latimer.

The officers for 1947–48 are: Director, Dr. D. E. South; Vice-Director, S. J. Jasper; Secretary-Treasurer, Dorothy Spragens; Librarian, Dr. A. A. Grau.

Kappa Mu Epsilon, Upsala College

A two-front action has been the long-range program of the activities of the *New Jersey Alpha* Chapter of *Kappa Mu Epsilon*. These are: the background of our present mathematics as brought about by men and events and the applications of mathematics in science and industry.

At the special midyear meeting the address was given by Professor Howard Fehr of Montclair State Teachers College. At the annual banquet and initiation meeting, Professor D. R. Davis, also of Montclair State Teachers College, spoke on *Mathematics of the Past and the Future*.

Students presented the following papers:

Pascal, by June Davidson

Abel, by Marjorie Cohen

Lobajevski, by Dagny Heggem

Certain theorems in non-euclidean geometries, by William Melchinger.

The officers for 1946–47 were: President, Dagny Heggem; Vice-President, and Treasurer, June Davidson; Secretary, Marjorie Cohen; Historian, Natalie Manno; Faculty Sponsor, Professor M. A. Nordgaard. The officers for 1947–48 are: President, Marjorie Cohen; Vice-President and Treasurer, June Davidson; Secretary, Frances Reischmuller; Historian, William Melchinger.

Mathematics Club, Mount Mary College

The activities of the *Mathematics Club* of Mount Mary College during the year were directed toward gaining a deeper appreciation of mathematics. The programs consisted of a series of reports on men who made major contributions in the field of mathematics.

After functioning as an independent organization, the club received recognition of its achievements by having four faculty members and sixteen club members inducted into *Kappa Mu Epsilon* on May 11, 1947. Mount Mary has the honor of becoming the *Wisconsin Alpha* Chapter of this fraternity.

The formal installation and initiation was presided over by Joseph J. Urbancek, faculty sponsor of the *Illinois Gamma* Chapter of the Chicago Teacher's College.

Officers for the year were: President, Dorothy Nilles; Secretary-Treasurer, Patricia Farrell; Faculty Advisor, Sister Mary Felice, S.S.N.D.

Mathematics Club, Cooper Union

After a period of inactivity due to the war, the Mathematics Club of Cooper Union reorganized this year. Students presented the following papers:

An introduction to the calculus of finite differences, by David Jagerman
Linear Diophantine equations, by Harry Hochstadt
Topology, by Eugene Wachspress
Geometry of the triangle, by Melvin Stern
Heaviside operational methods, by David Jagerman
Nomography, by Eugene Wachspress
Approximate summation of series, by David Jagerman
Methods of vector analysis, by Leon Nemerever.

Printed notes, prepared by the speakers, were distributed for the lecture on Heaviside operational methods and for the series of three lectures on methods of vector analysis. It was decided to continue this procedure at all future meetings. More material is often presented at a lecture than can be thoroughly absorbed by the members, and such notes, besides serving as a ready reference during the lecture, enable the members to review the subject at their leisure.

Officers for the year were: President, Leon Nemerever; Vice-President, David Jagerman; Secretary-Treasurer, Eugene Wachspress. Officers for 1947–48 are: President, David Jagerman; Vice-President, Harry Hochstadt; Secretary-Treasurer, Melvin Stern; Faculty Advisor, Professor J. N. Eastham.

Zeta Mu Tau, University of Washington

Zeta Mu Tau, mathematics honorary at the University of Washington, is composed of undergraduates from the departments of pure and applied mathematics, chemistry, physics, and engineering who have fulfilled a required grade-point average both in the major field and in the general university curriculum. The purpose of the organization is to stimulate interest in mathematical re-

search, particularly in the applied field. In addition to the annual initiation banquet for new members, three general meetings were held this year. Among the more significant speeches were:

Assumptions in engineering elasticity, by Marvin Stippes

A journey to France, by Professor R. M. Winger.

This year's officers included: President, B. R. Laverty; Vice-President, R. Nelson; Secretary, R. P. Kraft; Treasurer, W. R. Carter.

Mathematics Study Club, Immaculate Heart College

The papers delivered at the quarterly meetings were:

Postulates of arithmetic and application in proving $2+2=4$, by Miss Catherine Campion

Classification of numbers and proof of irrationality of $\sqrt{2}$, by Miss Sara Doherty

Zero and the use of l'Hospital's rule in $\frac{0}{0}$, by Miss Phyllis Beerling

History of theory of numbers, by Miss Suzanne Toulan

De Moivre's Theorem, by Miss Joan Pfisterer

Linear congruences, by Miss Sara Doherty.

The following book reviews were given:

A Mathematician's Apology, by Godfrey H. Hardy, reviewed by Miss Marian Snyder

The Magic of Numbers, by E. T. Bell, reviewed by Miss Mary Dominguez.

Five undergraduate students were guests of the Mathematical Association of America, Southern California Section, at the meeting held in March at Pomona College, Claremont, California. An enthusiastic report was brought back to the *Study Club* meeting which followed.

The year's activities culminated in May with a dinner given by the students in honor of Dr. Myrtie Collier, head of the Mathematics Department, who is retiring after seventeen years of successful service with the college.

This year's program was under the direction of: President, Phyllis Beerling; Vice-President, Lavada Moudy; Secretary, Margaret Wehr; Treasurer, Elizabeth Diedrich. Miss Beerling will serve again next year as President while the other offices will be filled at the first meeting in September.

Mathematics and Physics Club, University of Alberta

Papers presented to the *Mathematics and Physics Club* of the University of Alberta during the 1946–47 session were:

The early history of astronomy, by Mr. Albert Shaw

The Laplace transformation, by Professor Max Wyman

The theory of the chain-reacting pile, by Mr. George Kokatailo

The evaluation of π, by Miss Marion Roberts

The Bell helicopter, by Mr. K. Korsak

Bernoulli numbers, by Mr. Eoin Whitney

High speed flight, by Mr. Douglas Baines

Gas turbines and jet propulsion, by Mr. Ken Lobb

X-ray spectra, by Mr. Arthur Stephenson

Astro-physics, by Mr. Don Hall.

The year's activities ended with the annual banquet of the club held on March 17, 1947. The guest speaker was Mr. L. E. Gads, Assistant Professor of Engineering, whose topic was *The ordinary man's impression of mathematics and physics.*

The officers of the club for 1946–47 were: President, Miss Sydney Jones; Secretary-Treasurer, Severan Heiberg; Social Convener, Miss Marion Roberts; Executive members, Douglas Baines and Len Greenberg; Faculty Advisors, Professor G. O. Langstroth and Mr. R. C. Jacka.

The newly elected officers for 1947–48 are: President, Len Greenberg; Secretary-Treasurer, Miss Marion Roberts; Social Convener, Miss Lorna Boon; Executive members, Don Hall and Eoin Whitney; Faculty Advisors, Professor J. W. Campbell and Professor E. H. Gowan.

NEWS AND NOTICES

EDITED BY HARRY POLLARD, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to Harry Pollard, White Hall, Cornell University, Ithaca, New York.

INSTITUTE OF NUMERICAL ANALYSIS ESTABLISHED

Plans have been completed for the establishment of one of the newest units of the National Bureau of Standards, The Institute of Numerical Analysis, at the University of California at Los Angeles, according to an announcement by Dr. Edward U. Condon, Director of the Bureau.

One of the giant high-speed electronic computing machines, now under development by the Bureau of Standards, will be installed at the Institute when completed. These computers will solve problems in minutes that now take days to work out, and will solve in days problems that are now out of the reach of scientists. Design specifications call for high memory capacity and automatically sequenced mathematical operations from start to finish at speeds attainable only with electronic equipment.

The machines can conceivably revolutionize the field of applied mathematics. Of particular importance both to the physical sciences and to technical industries will be the fact that the Institute will be able to set up a mathematical counterpart of an actual situation, which permits the situation then to be studied through relatively inexpensive calculating rather than costly experimentation. Great as has been the progress of the past century, the time has come when many problems of great importance, especially in hydrodynamics, aerodynamics, and meteorology, can only be handled by computers working at speeds measured in millionths of a second.

The Institute has two primary functions. The first is research in applied mathematics aimed at developing methods of analysis which will extend the use of the high-speed electronic computers. The second is to act as a service group for western industries, research institutions, and government agencies. The service function will include not only the use of the machines for problem solving but also assistance in the formulation of problems in applied mathematics of the more complex and novel types. Service operations are to be initiated immediately, using the latest types of commercially available computing equipment.

PERSONAL ITEMS

Professor L. W. Cohen of the University of Kentucky has been appointed to an assistant professorship at Queens College.

Dr. Nancy Cole of Connecticut College has been appointed to an assistant professorship at Syracuse University.

Associate Professor H. A. Davis of West Virginia University has been promoted to a professorship.

Dr. Nathan Fine has been appointed to an assistant professorship at the University of Pennsylvania.

Professor H. D. Larsen of the University of New Mexico has been appointed to a professorship at Albion College.

Associate Professor A. N. Milgram of Notre Dame University has been appointed to an associate professorship at Syracuse University.

Dr. A. K. Mitchell has been appointed to an associate professorship at the University of Maryland.

K. H. Murphy of West Virginia University has been promoted to an assistant professorship.

I. D. Peters of West Virginia University has been promoted to an assistant professorship.

Dr. George Piranian of the University of Michigan has been promoted to an assistant professorship.

Dr. Murray Protter of Brown University has been appointed to an assistant professorship at Syracuse University.

Assistant Professor G. E. Schweigert of Purdue University has been appointed to an associate professorship at the University of Pennsylvania.

Dr. G. Tunell of the Carnegie Institute has been appointed to an associate professorship of geology at the University of California at Los Angeles.

The following appointments to instructorships are announced:

West Virginia University: Thomas Bauserman, Miss Marcia Saile
University of Rochester: Walter Klimzak

Professor Giacomo Albanese of the Normal School of Pisa and the University of Sao Paulo died June 8, 1947.

Professor Cora B. Hennel of Indiana University died on June 26, 1947.

Bernard Mason of Hofstra College died August 25, 1947.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirtieth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Wyoming, Laramie, Wyoming, on April 18 and 19, 1947. There were three sessions, with Professor Greta Neubauer of the University of Wyoming presiding at each.

There were sixty-four persons in attendance, including the following twenty-five members of the Association: C. F. Barr, D. L. Barrick, J. R. Britton, A. G. Clark, G. S. Cook, A. T. Craig, A. B. Farnell, H. T. Guard, Mrs. Leota C. Hayward, I. L. Hebel, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, A. E. Mallory, W. K. Nelson, K. L. Noble, O. H. Rechard, A. W. Recht, L. W. Rutland, Jr., Nathan Schwid, S. R. Smith, L. C. Snively, V. J. Varineau, Mrs. Lillie C. Walters.

At the business meeting the following officers were elected for the coming year: Chairman, H. T. Guard, Colorado State College of A. and M. A.; Vice-Chairman, I. L. Hebel, Colorado School of Mines; Secretary-Treasurer, J. R. Britton, University of Colorado. Invitations to meet at Colorado State College of A. and M. A. in 1948, and at Colorado School of Mines in 1949 were accepted.

The following papers were presented:

1. *Expansion of an arbitrary function in series of functions associated with Bessel functions*, by Professor Leonard Bristow, University of Wyoming, introduced by Professor C. F. Barr.

The author defined a set of functions by generalizing the Poisson integrals for Bessel and for Struve functions. For a suitable arbitrary function there was obtained an expansion resembling the generalized Schlomilch series.

2. *The solution of an integral equation*, by Professor W. H. Jurney, Colorado School of Mines, introduced by Professor I. L. Hebel.

3. *Note on functions of a matrix*, by Professor Clarence Ross, University of Denver, introduced by A. J. Lewis.

The matrix e^{kt} was expanded into a polynomial in k of degree not greater than $n-1$, where k is an $n \times n$ matrix. An application to the solution of linear homogeneous differential equations was explained.

4. *Bounds for the characteristic roots of a matrix*, by Professor A. B. Farnell, University of Colorado.

A brief history of this subject and related topics was presented. Let $\mathbf{A} = (a_{rs})$ be a square matrix of order n with complex numbers as elements. The equation $|\lambda \mathbf{I} - \mathbf{A}| = 0$, where $\mathbf{I}$ is the unit matrix and λ is a scalar, is called the characteristic equation of the matrix $\mathbf{A}$, and the roots λ_i, the characteristic roots. Several

new bounds for the characteristic roots were given. Let

$$\sum_s |a_{rs}| = R_r, \quad \sum_r |a_{rs}| = T_s, \quad \sum_s |a_{rs}| R_r = U_r, \quad \sum_r |a_{rs}| T_s = V_r.$$

Then $|\lambda|$ is not larger than any of the three numbers $\max_r (U_r)^{1/2}$, $\max_r (V_r)^{1/2}$, $\max_r (U_r V_r)^{1/4}$.

5. *A new method of approximating Fourier coefficients*, by G. L. Collins, Colorado School of Mines, introduced by Professor I. L. Hebel.

This speaker presented a simple method for evaluating the Fourier coefficients of a curve plotted to a predetermined scale. The essential idea of the method consisted of the use of a series of specially ruled transparent plastic sheets.

6. *Wallis' product for* π, by W. W. Mitchell, Jr., University of Colorado, introduced by Professor A. J. Kempner.

It was shown how Wallis determined the value of π between ever narrowing upper and lower bounds by a process of interpolation in a sequence of numbers related to the first quadrant areas under the curves $y=(1-x^2)^n$, $n=0, 1, 2, \cdots$.

7. *On complex roots of algebraic equations*, by Professor A. J. Kempner, University of Colorado.

Given an equation $f(z)=a_0z^n+\cdots+a^n=0$ with real coefficients and roots $z_k=x_k+iy_k$, $k=1, \cdots, n$, one knows how to establish by rational operations equations $G(x)=0$, and $H(y)=0$, each of degree n, such that each x_k is among the roots of the first, each y_k among the roots of the second equation. However, this leaves in each equation n^2-n roots unaccounted for. The location of these roots is determined by the theorem: The n roots of $G(x)=0$ are $x_j=\frac{1}{2}(z_k+z_l)$, $k, l=1, 2, \cdots, n$; the n roots of $H(y)=0$ are $y_j=\frac{1}{2}(z_k-z_l)$. A striking geometrical interpretation in the plane of complex numbers is possible.

Results are extended in toto to equations with complex coefficients without raising the degrees of $f(z)$, $G(x)$, $H(y)$ by letting $z=u+v$ with the restriction that with $u+v$, the number $u-v$ is also a root of $f(z)=0$. The function $G(x)$ is of the form $f(x)\cdot K^2(x)$, K being of degree $(n^2-n)/2$; $H(y)$ is of the form $y^nL(y^2)$, L being of degree $(n^2-n)/2$ in y^2. Similar results hold for the equation for r and for $e^{i\phi}$, $z=re^{i\phi}$.

8. *Statistical inference*, by Professor A. T. Craig, University of Iowa.

This paper was devoted to an exposition of the construction of a mathematical system adequate to furnish methods for drawing inferences from statistical data. The paper included an introduction to the Neyman-Pearson theory of testing statistical hypotheses.

9. *Is mathematics out of this world?* by Professor A. W. Recht, University of Denver.

The main thesis of this paper is that mathematics as presented in high schools and in colleges of liberal arts is out of this world in the sense that the principles of mathematics are set up in the classical and traditional way instead of in the

way in which they occur in real life. The suggestion is made that textbooks be written with the psychological approach by mathematicians who are also experts in fields of real application of mathematics. Problems should be presented as they occur in real life. It is only in this way that mathematics will be able to maintain the high reputation it has acquired in the atomic age; it is only in this way that students in the high schools and colleges will be kept interested in mathematics of reality, and not dazed by operations in a world of unreality.

10. *General mathematics*, by Professor Fred McCune, Colorado State College of Education.

In this paper the author asks why courses in "general" mathematics should duplicate training given in standard algebra and geometry courses. He believes that training in the fundamental skills of arithmetic is more important for the average secondary school student.

11. *The training of mathematics teachers*, by Professor K. H. Stahl, University of Colorado, introduced by the Secretary.

The attitude developed by students in mathematics has great influence not only on them, but also on us as teachers of mathematics. The teacher controls to a great extent the attitudes developed by members of the class, and it is therefore important that all teachers have a proper influence on their students. If the teacher himself is not well grounded in the material to be presented, it is quite unlikely that his influence will be wholesome. In all probability many persons become certified to teach in the elementary schools with very poor backgrounds in arithmetic. It is recommended that college teachers concern themselves with the mathematical preparation of elementary teachers.

12. *Report on the entrance requirement changes at the University of Colorado*, by Professor A. J. Kempner, University of Colorado.

Professor Kempner reported on the recent changes in entrance requirements for the Colleges of Arts and Sciences at the University of Colorado. All students must now offer three units of high school English, besides nine other units in "academic" subjects. These may not be selected arbitrarily; but students may enter the College without any high school work in any chosen one of the four large fields: foreign language, mathematics, natural sciences, social sciences. Under some arrangements students may even enter without any high school work in any chosen two of these fields.

There is opposition within the faculty to this scheme. Departments were not properly consulted.

In mathematics the situation is aggravated by the fact that a student who offers mathematics on his entrance requirements may substitute "high school arithmetic" and "general mathematics" for high school algebra and high school geometry which were required under the old rule.

Criticism of this last regulation centers around the fact that "general mathematics," as the term is understood in our part of the country, represents mathe-

matics courses which were introduced specifically for students who were either admittedly incapable of carrying the standard algebra and geometry courses, or who did not intend to go on to college training, but who wanted vocational courses in mathematics with a minimum of emphasis on theory and logical development. The department of mathematics refuses to recognize these courses as adequate prerequisites for college mathematics. These courses must not be confused with "unified mathematics" courses, which in some parts of the country go under the name of "general mathematics." For these, a strong case can be made out.

The mathematics department consulted groups of Colorado high school teachers, particularly mathematics teachers. The results were revealing. Over a hundred mathematics teachers of the Denver Section of C. E. A. protested unanimously against the changes. The Grand Junction Section, one of the other two sections in the state, sent a similar protest. The mathematics departments of two of the large Denver high schools, Denver North and Denver East, sent unanimous petitions to the president of the University, and so forth.

High school administrators generally favor the new rules, and regret that they do not go farther than they do. There exists scattered disapproval among them, but it has so far not become organized.

Our experience in Colorado proves that we have powerful allies among the high school teachers; they suffer and chafe under the steady deterioration of the standards and are, at least in Colorado, as a group more aware of the dangers and implications of the situation, and far more willing to fight for its improvement, than are college and university faculties.

In the lively discussion which followed the speaker's remarks, sentiment was opposed overwhelmingly to the elimination of mathematics as an entrance requirement, and as bitterly opposed to the admission of high school arithmetic and "general mathematics" in place of algebra and geometry.

J. R. BRITTON, *Secretary*

APRIL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-fourth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Mississippi Southern College, Hattiesburg, Mississippi, on Friday and Saturday, April 25 and 26, 1947. Professor W. V. Parker, Chairman of the Section, presided at the Friday afternoon and Saturday morning sessions. Professor W. L. Johnson, Vice-Chairman for Mississippi, presided at the joint dinner with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics.

The attendance was sixty-five including the following thirty members of the Association: T. A. Bickerstaff, H. E. Buchanan, Margaret R. Davis, W. L. Duren, Jr., L. M. Garrison, F. C. Gentry, A. Gilmore, W. C. Griffith, W. L. Johnson, H. T. Karnes, C. G. Killen, Z. L. Loflin, Dorothy McCoy, A. C. Maddox, B. E. Mitchell, S. B. Murray, I. C. Nichols, W. V. Parker, P. K. Rees, F. A. Rickey,

H. F. Schroeder, Maurice Singer, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, P. M. Tullier, Jr., Maralena White.

The officers elected at the business meeting of the section were as follows: Chairman, W. L. Johnson, Mississippi Southern College; Vice-Chairman for Mississippi, T. A. Bickerstaff, University of Mississippi; Vice-Chairman for Louisiana, A. L. Loflin, Southwestern Louisiana Institute; Secretary-Treasurer, F. C. Gentry, Louisiana Polytechnic Institute. The next meeting will be held in February or March, 1948, at Southwestern Louisiana Institute, Lafayette, Louisiana.

Professor H. E. Buchanan, Tulane University, was invited to give the principal address at the joint dinner. His topic was *Mathematics Teaching—Tulane Brand*. Professor H. L. Smith, Louisiana State University, was invited to speak at the Saturday morning session. His subject was *A Foundation for the Point Calculus of Grassmann*. The Friday afternoon session was devoted to short papers. Abstracts of all these papers follow:

1. *A foundation for the point-calculus of Grassmann*, by Professor H. L. Smith, Louisiana State University.

The Grassmann point calculus may be based on a foundation consisting of seven postulates involving an unspecified field of numbers and the undefined terms point, scalar product $s(p_0, p_1; p_2, p_3)$ of a point pair (p_0, p_1) by a point pair $p_2, p_3)$, and the point $\bar{p}$ conjugate to a point p. The resulting theory is valid in Hilbert, as well as in Euclidean, spaces.

2. *Installment buying*, by Professor I. C. Nichols, Louisiana State University.

3. *Mathematics applied to meteorology*, by Mr. J. T. Lee, Mississippi Southern College, introduced by Professor Johnson.

This paper was a short introduction to the application of mathematics to meteorology and weather forecasting. A general development of several of the basic equations, such as the gradient wind formula and the fundamental concepts of meteorology was included.

4. *Problems concerning volumes*, by Professor C. D. Smith, Mississippi State College.

The cone inscribed in a sphere whose volume is equal to the volume of the segment of the sphere cut off by its base was found to have a volume approximately equal to that of the inscribed cone of maximum volume. A similar comparison was made for cones inscribed in ellipsoids of revolution.

5. *Three cubic loci*, by Professor F. C. Gentry, Louisiana Polytechnic Institute.

It was shown that if a variable point P of the plane traces the cubic of Darboux relative to a given triangle, the joins of the vertices of the pedal triangle of P and the corresponding vertices of the given triangle are concurrent in a point Q on the cubic of Lucas. If P traces the 17-point cubic, the perpendiculars from the harmonic associates of P on the corresponding sides of the given triangle are concurrent in a point R on the cubic of Darboux, and the joins of their feet and

the corresponding vertices of the given triangle are concurrent in a point S on the cubic of Lucas.

6. *Discussion of the report of the Association Committee for the Coordination of Studies on Mathematical Education*, by L. M. Garrison, Louisiana Polytechnic Institute; T. A. Bickerstaff, University of Mississippi; Z. L. Loflin, Southwestern Louisiana Institute.

F. C. GENTRY, *Secretary*

APRIL MEETING OF THE SOUTHWESTERN SECTION

The seventh annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico in Albuquerque on April 4, 1947. Professor E. A. Hazlewood, Chairman of the Section, presided at the afternoon session. Professor H. D. Larsen presided at the banquet at which President W. M. Whyburn of Texas Technological College was the guest speaker.

The attendance was twenty-five including the following thirteen members of the Association: L. M. Bauer, J. H. Butchart, Lincoln La Paz, H. D. Larsen, B. D. Roberts, H. P. Rogers, Arthur Rosenthal, Annie N. Rowland, F. W. Sparks, R. S. Underwood, Earl Walden, R. L. Westhafer, W. M. Whyburn.

At the business meeting the following officers were elected: Chairman, H. D. Larsen, University of New Mexico; Vice-Chairman, R. F. Graesser, University of Arizona; Secretary-Treasurer (four years), B. D. Roberts, Highlands University; Governor (three years), R. S. Underwood, Texas Technological College. These officers also constitute a committee to choose the next traveling lecturer sponsored by the Southwestern Section. It was voted to hold the 1948 meeting of the Section in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science, providing the latter meeting is held within the geographical limits of the Section.

The program consisted of the following papers:

1. *Vector methods in modern geometry*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

Professor Butchart called attention to vector proofs of theorems in modern plane geometry, especially those concerning centroids. He also recommended the use of the scalar product of two vectors to prove theorems in modern geometry involving projections or the squares of distances.

2. *An inverse variation problem*, by Frank Lane, University of New Mexico, introduced by Professor Lincoln La Pas.

Darboux's result that the integral curves of the differential equation $y''=f(x, y, y')$ can be regarded as the extremal system of a non-singular variation problem, $I=\int_{x_1}^{x_2} f(x, y, y')\,dx=$ minimum, cannot be extended to 3-space. The first example of a pair of differential equations

$$y'' = F(x, y, z, y', z'), \qquad z'' = G(x, y, z, y', z') \tag{1}$$

of which the integral curves are not the extremals of any non-singular variation problem of the form

$$(2) \qquad I = \int_{x_1}^{x_2} f(x, y, z, y', z')dx = \text{minimum},$$

was given by Lincoln La Paz in 1928. Additional isolated examples of such systems of differential equations have since been given by other investigators.

Mr. Lane exhibited classes of differential equations (1), the right members of which involve arbitrary functions of y' and z', no one of which has integral curves that are the extremal system of a non-singular variation problem of the form (2). Because of the freedom of choice thus provided in selecting the functions F and G, it was possible to so choose these functions that the resulting system of differential equations could be integtated explicitly. Certain properties of the resulting non-extremal, four-parameter families were discussed.

3. *Extended analytic geometry applied to Diophantine equations*, by Professor R. S. Underwood, Texas Technological College.

Professor Underwood used a method suggested by the three-axes case of his extended analytic geometry (this MONTHLY, May, 1945) to solve certain Diophantine equations. In particular, the speaker showed how the solution of two simultaneous linear equations in three unknowns is reduced readily to the solution of a single equation in three unknowns.

4. *On the determination of optimum flight paths*, by Professor Morris Hendrickson, University of New Mexico, introduced by the Secretary.

Professor Hendrickson discussed the problem of determining the route which should be flown through a wind field varying in time and space in order to get from a point A to a point B in the minimum time, assuming that the flight takes place at a constant altitude and with constant air speed. The problem for a flat surface has been treated in the literature by Zermelo who derived both necessary and sufficient conditions, and for an n-dimensional euclidean space by Levi Civita who derived only necessary conditions. The speaker discussed the problem of flight over a spherical surface which he had investigated jointly with G. E. Forsythe and K. J. Arrow. A necessary condition was derived in the form of a differential equation which the minimal path must satisfy, and a practical method for carrying out the approximate integration of this equation was presented.

5. *Derivatives of fractional order*, by C. P. Stroud, Highlands University, introduced by Professor B. D. Roberts.

Mr. Stroud, a student at Highlands University, discussed derivatives obtained from the form for successive differentiation

$$\frac{d^n}{dx^n}(x^m) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

by extending n to fractional, irrational, and negative values. Various formulas for such generalized differentiation were presented. The speaker also suggested the use of fractional indices for operations other than differentiation.

H. D. LARSEN, *Secretary*

APRIL MEETING OF THE IOWA SECTION

The thirty-fourth annual meeting of the Iowa Section of the Mathematical Association of America was held at the Iowa State Teachers College in Cedar Falls on Friday and Saturday, April 18 and 19, 1947, in conjunction with the Iowa Academy of Science. Professor L. W. Swanson, Chairman of the Section, presided.

The attendance was forty-five, including the following thirty members of the Association: E. W. Anderson, J. W. Beach, E. L. Canfield, E. W. Chittenden, N. B. Conkwright, W. M. Davis, R. E. Gaskell, B. E. Gillam, Cornelius Gouwens, J. J. L. Hinrichsen, D. L. Holl, L. A. Knowler, O. C. Kreider, R. J. Lambert, R. B. McClenon, J. V. McKelvey, Martha M. McKelvey, C. J. Maloney, E. N. Oberg, H. V. Price, Fred Robertson, W. J. Rusk, W. M. Stone, L. W. Swanson, H. P. Thielman, H. C. Trimble, Henry Van Engen, Roscoe Woods, C. C. Wylie, E. A. Zubay.

The following officers were elected for the coming year: Chairman, Professor H. P. Thielman, Iowa State College; Secretary, Professor Fred Robertson, Iowa State College.

The first six of the following papers, including the invited hour address by Professor Holl, were read at the Friday afternoon session. The remainder were read Saturday morning.

1. *On differential difference equations*, by Professor H. P. Thielman, Iowa State College.

The equation $f'(x+a)=K(x)f(x)$ was considered, when a is a constant and $f(x)$ is an unknown real function. This equation was shown to be equivalent to an integral equation of the Volterra type with a discontinuous kernel. Analytic solutions were given for the particular case in which $K(x)$ is a constant, say k. For certain values of k the given solutions were either monotone, or periodic functions.

2. *Some applications of the finite Fourier transformations*, by Professor R. E. Gaskell, Iowa State College.

Use of the finite Fourier transformation in solving boundary value problems is limited to problems leading to special linear differential operators, and involving boundary conditions of a special form. For example, if the finite Fourier sine transformation is to be used, only values of the unknown function and its even derivatives (with respect to the transformed variable) may be involved in the boundary conditions. A problem was given showing how this requirement can be relaxed if more general transformations are used.

3. *On the number of paths in a finite partially ordered set*, by Professor E. W. Chittenden, State University of Iowa.

4. *Dirichlet's problem*, by Professor D. L. Holl, Iowa State College.

The first boundary problem of potential theory, Dirichlet's problem, is to find a function harmonic in a region and taking on a preassigned value on its boundary. By Green's identities many properties of such a function can be established. It was shown that the problem is equivalent to finding Green's function.

Gauss, Kelvin, Dirichlet and Riemann gave faulty proofs of the existence of a harmonic function by considering it as a minimal problem in calculus of variations. Hilbert was first to specify proper conditions for this problem as a minimal problem. Direct methods of Ritz, and Trefftz, approximation by finite differences, and experimental models by soap film surfaces were discussed.

5. *A model for irreducible double modules*, by Professor Bernard Vinograde, Iowa State College, introduced by the Secretary.

A model was given which included the irreducible double module $V=L\times R$, x in V, such that $L=\sum l_i L_0$, and $R=\sum R_0 r_i$ with $R_0\cong L_0$, where L and R are division rings. The construction depends on the existence of a representation of L over R.

6. *Solution of iterated amplifiers by generalized Laplace transform*, by W. M. Stone, Iowa State College.

A simultaneous system of difference equations arising from an iterated linear amplifier system discussed by Faust and Beck (*Journal of Applied Physics*, vol. 17, pp. 749–756) was solved by means of the generalized Laplace transform. With this method there was no necessity for eliminating arbitrary constants or making assumptions as to the nature of the solution.

7. *On the general theory of functions*, by Professor E. W. Chittenden, State University of Iowa.

This paper was read by title.

8. *Nearly efficient estimates of variance components*, by S. Lee Crump, Iowa State College, introduced by the Chairman.

For statistical data classified into groups of unequal sizes, the efficient (maximum likelihood) estimates of the variance components are extremely difficult to compute. Estimates, based on Newton's method of solving equations, which are very nearly fully efficient, were presented.

9. *A property of the projective cubic*, by C. J. Maloney, Iowa State College.

The set of all tangents to a conic is called a line conic. The lines of the line conic cut any two fixed lines in projective ranges. If a projective relation is set up between a (first order) pencil of lines and the lines of the line conic by means of a projective range on some one line of the line conic, the intersections of corresponding rays will trace a cubic, called from the manner of its generation a projective cubic. It was shown geometrically that the cubic so generated is always unicursal, and that it is crunodal, cuspidal, or acnodal, according as the center of the pencil is outside, on, or inside the conic.

10. *Curve fitting—an art or a science*, by Professor G. W. Snedecor and Professor G. W. Brown, Iowa State College.

This paper was read by title.

11. *Interpretation of "college preparation" by individual teachers of high school mathematics*, by Professor H. C. Trimble, Iowa State Teachers College.

It was contended that the words and actions of college people convince teachers of high school mathematics that: (1) Certain topics are essential for the college preparatory student; (2) The fields of arithmetic, algebra, and geometry

must be taught separately; and (3) Standardized tests are a valid measure of a student's preparation for college mathematics.

After a brief discussion of the consequences of these convictions, three questions were raised, namely: (1) What *do* we in the colleges want? (2) How can we make our wants known to teachers of high school mathematics? and (3) What assurance can we give to high school teachers that their students will get a fair chance to show what they know as they enter college?

The writer's own answers to these questions were stated, and a plea was made for further study and action to clarify the situation.

FRED ROBERTSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pa., November 22, 1947
ILLINOIS
INDIANA
IOWA, Fairfield, April 16–17, 1948
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., December 6, 1947
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST, Eugene, Oregon, March, 1948
PHILADELPHIA, Bryn Mawr, Pa., November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Schenectady, N. Y., May 1, 1948
WISCONSIN, Beloit, May 8, 1948

WILLIAM L. HART'S

Texts on College Algebra

NOW READY

Brief College Algebra

REVISED EDITION

This book was written for the well-prepared student who needs at most only a relatively brief review of intermediate algebra and who deserves the opportunity of reaching the interesting parts of college algebra quickly. BRIEF COLLEGE ALGEBRA presents a concise, but logically complete review, followed by a normal leisurely treatment of all usual topics of college algebra. It offers an integrated body of material dealing with logarithmic graphing, the method of least squares, also related topics in curve fitting and the algebra of statistics. Text pages: 299

NOW READY

Introduction to College Algebra

REVISED EDITION

A complete, mature presentation of intermediate algebra, followed by a minimum treatment of the most essential chapters of college algebra through the stage of determinants and probability. INTRODUCTION TO COLLEGE ALGEBRA is designed as the basis of a one-semester course for college freshmen who should cover intermediate algebra and topics from college algebra. Text pages: 272

AVAILABLE—FREQUENT PRINTINGS

College Algebra

THIRD EDITION

Presents a comprehensive treatment of the usual content of college algebra, plus various supplementary topics, preceded by a complete presentation of intermediate algebra. For a homogeneous class with either two or three semesters of preparation in secondary algebra, there is an appropriate starting point in this book. Also included is a substantial amount of supplementary material of interest in experimental fields and statistics. Text pages: 362

Stock of previous editions of above books is still available.

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

PLANE TRIGONOMETRY

REVISED EDITION

By *FRED W. SPARKS* and *PAUL K. REES*

For over ten years a basic first-year text, this book is now even clearer, simpler and more efficient. Graded exercises for each lesson offer a total of 1,350 brand-new problems in aerial navigation, mechanics, engineering and other topics of vital interest to students today. Other notable improvements include:

- New discussion of significant figures.
- Simplified approach to the characteristic of the logarithm.
- Improved explanations of inverse trigonometric functions, equations, etc.
- Over 40 newly drawn figures to clarify basic principles.

6 x 9 in., 255 pp.

MATHEMATICS OF FINANCE

SECOND EDITION, REVISED AND ENLARGED

By *THOMAS M. SIMPSON, ZAREH M. PIRENIAN* and *BOLLING H. CRENSHAW*

Here is the best possible preparation for students intending to enter business. For greater teaching convenience, the text is arranged in two parts, which may be obtained either in a single volume, or in two volumes for single-semester courses.

Note these features:

- 2568 problems; enlarged tables.
- One entire chapter on statistics.
- Simplified treatment of general annuities.
- Enlarged treatment of life insurance and problems of amortization of all types of debts.

6 x 9 in., 456 pp.

Send for your approval copies today

PRENTICE-HALL, INC., 70 FIFTH AVENUE NEW YORK 11, N. Y.

Two highly significant new texts—

COLLEGE ALGEBRA

By **MOSES RICHARDSON**, author of FUNDAMENTALS OF MATHEMATICS; Former Member of Institute for Advanced Study at Princeton.

This recently published, remarkably lucid exposition of College Algebra

- gives unusual insight into sound mathematics, correcting traditional mistakes.
- is far more complete than most texts:
  beginning with a full review of elementary algebra on a mature level, it covers not only all conventional subjects, but adds optional material now in great demand by science students—and by mathematics students preparing for future specialization;
- stresses fundamental concepts, but not at expense of technical achievement;
- includes ample exercises, a great number with practical application.

6 x 9 in., 472 pp.

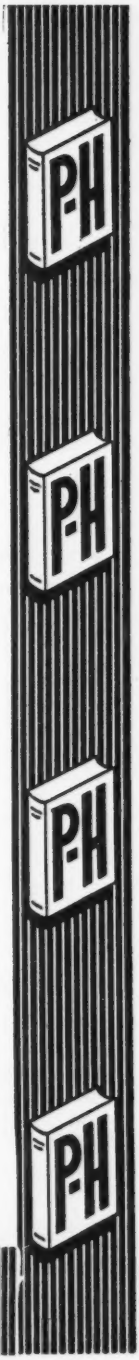
ADVANCED CALCULUS

By *DAVID V. WIDDER*

This superb new text is distinguished by two gratifying innovations in subject matter: chapters on the Stieltjes Integral and the Laplace Transform. These subjects have never before been easily accessible in elementary form, and they are becoming increasingly important to mathematicians. Widely known for his work on the Laplace Integrals and Transform during his past fifteen years at Harvard, the author presents not only a fine exposition of the more usual topics, but presents an unusually clear treatment of Line Integrals and Green's Theorem.

6 x 9 in., 432 pp.

Send for your approval copies today



PRENTICE-HALL, INC., 70 FIFTH AVENUE NEW YORK 11, N. Y.

NEW MATHEMATICS TEXTS

COLLEGE ALGEBRA

By E. RICHARD HEINEMAN

Associate Professor of Mathematics, Texas Technological College

This text for freshmen contains a thorough review of high-school algebra with more than 2800 carefully graded problems. Common errors are discussed to give the student an understanding not only of the correct methods, but also of the error in each incorrect operation. A simplified definition of a function and a logical treatment of graphic solutions are provided. The new characteristic rule for logarithms is used and the old rule is listed as an alternative.

To be published in the fall. $3.00 (probable)

INTERMEDIATE ALGEBRA

By UNDERWOOD, NELSON AND SELBY

R. S. Underwood is Professor of Mathematics, Texas Technological College; T. R. Nelson is Associate Professor of Mathematics, Agricultural and Mechanical College of Texas; S. Selby is Professor of Mathematics and Head of the Department, University of Akron.

Presupposing one year of high-school mathematics, this text presents a terminal course in algebra as well as a foundation course for students preparing for more advanced college mathematics. Extremely simple in style, clear and concise, the book maintains a consistent, upward-sloping level of difficulty. There are many illustrative examples and exercises, carefully chosen to cover typical cases.

To be published in the fall. $2.60 (probable)

ELEMENTARY CONCEPTS OF MATHEMATICS

By BURTON W. JONES

Professor of Mathematics, Cornell University

For students not intending to specialize in mathematics or science, this text is designed to provide a firmer understanding of what mathematics they have had and to supply additional training of both use and interest. It clarifies the everyday mathematical concepts often only dimly understood: compound interest, the graph, averages, probability and games of chance, cause versus coincidence. Puzzle material and many exercises are an important part of the book.

To be published in the fall. $4.00 (probable)

THE MACMILLAN COMPANY • 60 Fifth Avenue • New York 11

5 New Wiley Books!

NOMOGRAPHY

By ALEXANDER S. LEVENS, *Associate Profesor of Mechanical Engineering,* University of California at Berkeley

Here is a time-saving, accurate book which shows you short-cut solutions to difficult problems of engineering, statistics, or business by means of nomographs. It presents the basic information on the theory and construction of charts involving straight line scales, curved scales and combinations for these elements. The alignment chart is especially stressed. The book is crammed with interesting examples.

Ready in December **Approx. 161 pages** **Prob. price $3.00**

INTRODUCTION TO THE THEORY OF EQUATIONS

By LOIS W. GRIFFITHS, *Associate Professor of Mathematics,* Northwestern University

Designed for students with a background in differential calculus, this book introduces the theory of equations, determinants, and matrices. A book that has been tested in the classroom, its logical exposition makes it suitable for advanced or elementary levels.

1947 **278 pages** **$3.50**

ELEMENTARY NUCLEAR THEORY

A Short Course on Selected Topics

By H. A. BETHE, *Professor of Phyics,* Cornell University

An introduction to nuclear theory, this book treats the subject from an empirical point of view. Dr. Bethe analyzes nuclear forces, the central problem of nuclear physics, and explains the available evidence in terms of the behavior of the simplest nuclear systems.

September, 1947 **147 pages** **$2.50**

SEQUENTIAL ANALYSIS

By ABRAHAM WALD, *Professor of Mathematical Statitics,* Columbia University

The first book-length treatment of an important and recently-developed method of statistical inference. Highly understandable, the book covers the general theory of the sequential probability ratio test, its applications, and an introduction to problems of sequential multi-valued decisions and estimates.

1947 **212 pages** **$4.00**

INTRODUCTION TO MATHEMATICAL STATISTICS

By PAUL G. HOEL, *Associate Profesor of Mathematics,* University of California at Los Angeles

This book is designed to give the reader a comprehensive introduction to the theory and applications of modern statistical methods. Such topics as these are included: sampling inspection, non-parametric methods, two types of errors, and sequential analysis.

1947 **256 pages** **$3.50**

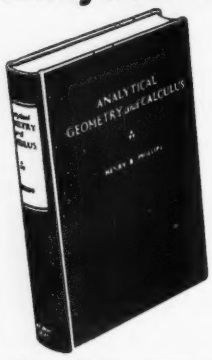
JOHN WILEY & SONS, Inc., 440 Fourth Ave., New York 16, N.Y.

Analytical GEOMETRY and CALCULUS

by

HENRY BAYARD PHILLIPS

Formerly Head of the Department of Mathematics
Massachusetts Institute of Technology



457 pp, index, answers to problems, 269 figures; plus 64 pp supplementary problems and answers; stiff cloth cover

per copy $6.00

An excellent textbook, designed in the form and order in which analytical geometry and calculus are required for courses in science and engineering.

Certain fundamental topics in analysis needed for the logical development of calculus are discussed in the first chapter. An elementary treatment of vector analysis and functions of a complex variable is included for work in mechanics and for the analysis of various physical fields.

CONTENTS

Limits and Continuity; Derivative and Differential; Integration and Summation; Algebraic Equations and Graphs; Determinants; Trigonometric Functions; Exponential and Logarithmic Functions; Parametric Equations; Polar Coordinates; Vectors; Formulas and Methods of Integration; Further Applications of Integration; Series with Real Terms; Complex Numbers; Functions of a Complex Variable; Space Coordinates and Vectors; Partial Differentiation; Multiple Integration.

ADDISON-WESLEY PRESS INC., Cambridge 42, Mass.

BOOK NEWS

ANALYTIC GEOMETRY, Revised Edition

By Raymond W. Brink

FOR a rich, complete, and adaptable course in analytic geometry, this is an ideal text. Throughout the emphasis is on logic and method. Problems, carefully graded and distinctive for their variety, freshness, and usefulness, provide for immediate application of theory, and formal exercises give training in the development of technique. The book is particularly useful as preparation for calculus and other advanced mathematical studies. 8vo, 350 pages. $2.90.

ESSENTIALS OF ANALYTIC GEOMETRY

By Raymond W. Brink

SIMPLER, more flexible, and somewhat less detailed than the author's *Analytic Geometry*, this book covers fully all topics usually taught in analytic geometry courses in American colleges and universities. It is compact, well-organized, clearly written, and provides sufficient material in method, information, and problems for preparation for calculus and for the development of general mathematical maturity and background. Small 8vo, 233 pages. $2.60.

D. APPLETON-CENTURY COMPANY

35 West 32nd Street New York 1, New York

Timely Books from McGraw-Hill

COLLEGE ALGEBRA

By Frederick S. Nowlan, University of British Columbia. 401 pages, $3.00

• A comprehensive text for college freshmen. Distinguished by a more detailed and careful review of elementary material than is usual, and by the completeness with which the subject matter is developed, the book is both mathematically sound and easily understandable.

ELEMENTARY DIFFERENTIAL EQUATIONS. *New third edition*

By Lyman M. Kells, U. S. Naval Academy. 333 pages, $3.00

• A thorough revision of a standard textbook especially suitable for students of engineering and applied sciences. Nearly every article has been clarified, expanded, and strengthened. Many new problems have been added.

HIGHER MATHEMATICS FOR ENGINEERS AND PHYSICISTS

By I. S. Sokolnikoff, University of California at Los Angeles, and E. S. Sokolnikoff. Second edition. 587 pages, $5.00

• Introduces the student to those branches of mathematics most frequently encountered by the engineer in his practice and by research specialists in the applied sciences.

SAMPLING INSPECTION

By the Statistical Research Group, Columbia University. OSRD. 395 pages, $5.25

• Gives a systematic account of certain of the best current inspection practices, together with tables and detailed instructions for carrying out these practices.

SELECTED TECHNIQUES OF STATISTICAL ANALYSIS

By the Statistical Research Group, Columbia University. OSRD. 440 pages, $6.00

• Deals with a series of problems which occur frequently in planning, analyzing, or interpreting quantitative data. Various techniques appropriate to these problems are explained in terms of both general principles and specific procedures.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

by BRITTON & SNIVELY

University of Colorado

Algebra for College Students

(529 pages—$3.00)

With major stress on underlying ideas, this text includes a refresher course in the fundamentals of arithmetic and elementary algebra as well as topics customarily covered in college courses. An abundance of review and graded exercises are offered. Published in April, 1947, the book has already been widely adopted. Among colleges using this text are:

Bates College
Brigham Young University
Clark University
Franklin & Marshall College
Iowa State College
James Millikin University
Johns Hopkins University
Marquette University
University of Colorado
University of Denver
University of Kentucky
University of Michigan

Intermediate Algebra

(Published September 9th—337 pages—$2.00)

. . . includes the first twelve chapters of

Algebra for College Students

and additional chapters on: *Logarithms, Progressions and the Bionomial Theorem, and Systems Involving Quadratic Equations.*

. . . contains a thorough review of the fundamental ideas and techniques of arithmetic and algebra for review and reference.

. . . is an up-to-date text offering statements of actual operations performed in solving problems—a system supplanting misleading catch-words and phrases.

. . . the text is supplemented by tables and carefully graded exercises both oral and written.

Rinehart & Company, Inc.

232 MADISON AVENUE · NEW YORK 16, N. Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

10

HNOLOGY DEPARTMENT

LIBRARY
JAN 8 - 1948
DETROIT

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 10

PART I

CONTENTS

Mathematics for Liberal Arts Students . . C. B. ALLENDOERFER 573
Geometry of the Kasner Triangle . . . . G. H. F. GARDNER 579
Nonseparable Convex Systems . . . . . H. HADWIGER 583
An Inversion Formula for an Integral Related to Dirichlet Series . . . . . W. C. G. FRASER 586
Mathematical Notes . . L. F. TOTH, N. J. FINE, H. P. EVANS 589
Classroom Notes. . . . . R. A. JOHNSON 594
Elementary Problems and Solutions 596
Advanced Problems and Solutions 601
The Mathematical Association of America 605
New Members. 605
The Twenty-ninth Summer Meeting of the Association . . . 608
Calendar of Future Meetings 613
Recent Publications 613
Clubs and Allied Activities 616
News and Notices 619
Index to Volume 54, 1947 620

DECEMBER 1947